Speculations on Physical Discretization and Arithmetic Geometry

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Abstract

We present some speculations concerning quantum systems in which there is a discretization in the values of fields and the spacetime due to the presence of a cutoff in the target space. This can be viewed as specifying a quantum theory in which the reduced Planck constant $\hbar$ satisfies the relation $2\pi\hbar = N$ with N a positive integer greater than one. Number theoretic structures such as finite fields and schemes in characteristic p enter in a structural way, and can be packaged in the language of arithmetic geometry. The resulting effective field theories have a truncated spectrum of independent higher dimension operators, in line with Swampland considerations. The associated geometries can be interpreted as building up a physical system from quantum error correcting codes. We also present some physically motivated conjectures connected with this construction. Using the developed formalism, we take some first steps in analyzing the geometry associated with quantized Fayet-Iliopolous parameters. We also propose a relation between supersymmetric indices and the Hasse-Weil Zeta function of schemes in characteristic p as well as a characteristic p analog of geometric engineering, including a conjectural correspondence between the moduli space of Calabi-Yau varieties of ADE singularities and topological field theories. In a suitable large N limit, we also observe the emergence of more fine-grained topological features.

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1 Introduction

There is a seductive appeal to discretizing the laws of Nature. That being said, our best understanding of fundamental physics continues to make heavy use of continuum concepts.

In this note we argue that in some quantum systems with a cutoff, there is a natural formulation in terms of structures which appear in number theory and in particular arithmetic geometry. This algebro-geometric language provides a way to transport many features of smooth geometry to a discretized setting. Our discussion will necessarily be somewhat speculative, but hopefully this will not distract too much from the main contours of the proposal.

To keep our analysis well-defined, we shall mainly focus on situations in which the spacetime as well as the target space for our fields are discretized in some way. This sort of situation arises in many physical situations. For example, an experimentalist may only be able to probe a system at a minimal time interval $t_{\min}$, and moreover, the values that are recorded by their measuring device may also be limited to some finite discretized level of approximation. At a more ambitious level, one might consider formulations of quantum gravity in which there is a minimal Planck scale for measurements.

Of course, some immediate issues with studying these sorts of systems is that lattice formulations of quantum theories tend to break most spacetime symmetries (such as Lorentz invariance), and extreme fine-tuning is often required to recover these structures at long distances. Similar issues are often present in non-commutative deformations of spacetime as well as matrix model approaches to quantum gravity. In the arithmetic context, however, there are analogs of the Lorentz group which can be maintained even in the discretized setting.

Our operating assumption will be that once we discretize the target space and spacetime, there is a natural sense in which the number of quanta which can be packed into any region of the target space is discretized. For example, in the case of a bosonic scalar field theory, we can interpret this in terms of the standard prescription for computing operator correlation functions via the path integral through expressions such as:

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle = \frac{\sum\limits_{\phi} [d\phi] \exp\left(iS[\phi]/\hbar\right) \mathcal{O}_1 ... \mathcal{O}_n}{\sum\limits_{\phi} [d\phi] \exp\left(iS[\phi]/\hbar\right)}\,, \tag{1.1}$$

but in which the fields ϕ range over a discrete set such as the integers, and the parameter $\hbar$ satisfies the condition:

$$\hbar = \frac{N}{2\pi}, \tag{1.2}$$

with $N > 1$ a positive integer. In this setting, evaluating the action on any field configuration results in $S[\phi]$ an integer, but in which only its value modulo N actually matters. One of our aims will be to show how to extend this sort of observation to the standard set of fields

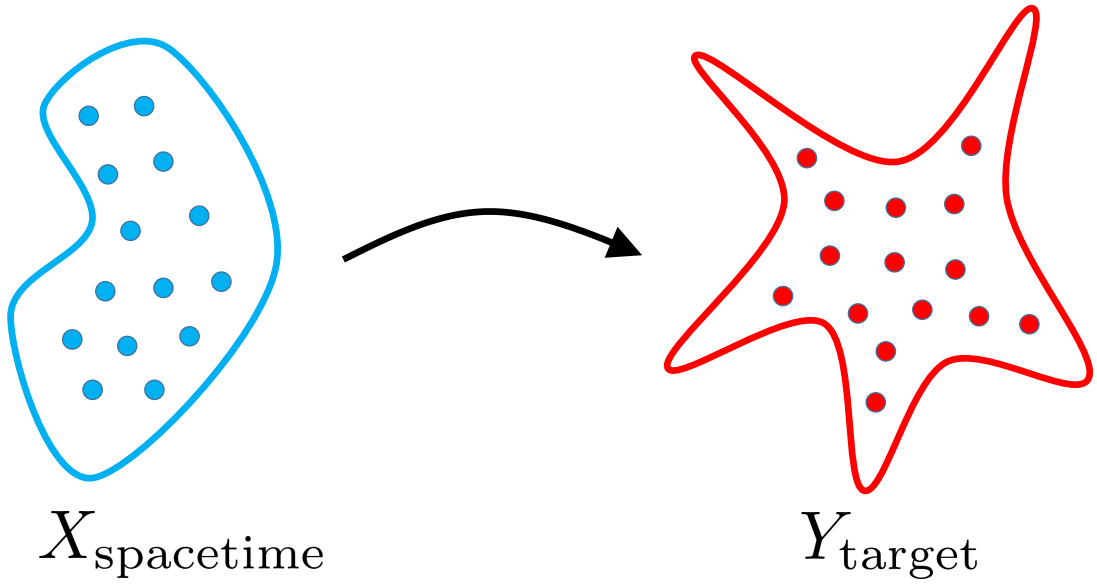

Figure 1: Depiction of a map between two varieties in characteristic p. Each dot indicates a point of the corresponding discretized geometry. These maps can be viewed as field configurations in a sigma model, with $X_{\text{spacetime}}$ the spacetime and Y_{target} the target space.

encountered in quantum field theory, including fermions, gauge fields, and even gravitons. We will also argue that supersymmetry still makes sense.

As one might expect, additional features become manifest if we restrict to the special case where $2\pi\hbar = p$ a prime number. When we do so, we can borrow much of the apparatus of algebraic geometry in characteristic p to formulate and study the resulting physical systems.[1] Again by way of example, we will show that bosonic scalars of the discretized field theory can be viewed as specifying a map between schemes defined in characteristic p:

$$\phi : X_{\text{spacetime}} \to Y_{\text{target}}. \tag{1.3}$$

Of course, the sense in which the "spacetime" of the field theory makes sense in characteristic p is that it is a suitably discretized space, and we can equip it with a topology, and non-trivial sheaves and maps to other spaces. See figure 1 for a depiction.

It is also clear that some cherished physical structures such as a notion of distance as defined by a metric will necessarily fall by the wayside in the arithmetic setting. There are, however, close analogs which retain much of the physical flavor present in Riemannian geometry in characteristic zero. For example, we can consider the space of symmetric bilinear forms specifying maps $T^*X \otimes T^*X \to \mathbb{F}_p$. This is the characteristic p analog of the graviton. Indeed, our formulation will be flexible enough to demand that we only work with struc-

[1] We review some aspects of arithmetic in characteristic p later on.

tures invariant under suitable coordinate redefinitions, as captured by morphisms between schemes.

Another aim of our analysis will be to study the impact on effective field theories generated in this way. We find that the spectrum of higher dimension operators which can enter the effective action are automatically truncated. This in some sense follows from the fact that in modular arithmetic, we have Fermat's little theorem, which tell us that for integers $m \in \mathbb{Z}$, reduction modulo p a prime always satisfies $m^p = m \bmod p$. Applied to a power series expansion in a physical field, this automatically leads to a truncated effective action. This is in line with some Swampland considerations such as [1, 2] which suggest the appearance of correlated Wilson coefficients in any quantum field theory coupled to higher dimension operators.

We envision applying these sorts of considerations to the study of quantum gravitational systems. For example, our physical systems in characteristic p naturally produce examples of quantum error correcting codes, and this has been a topic of some interest in the holographic quantum gravity literature (see e.g. [3–5]). As another example, consider the analog of supersymmetric field theory with a quantized Fayet-Iliopoulos (FI) parameter. There are hints from supergravity (see e.g. [6–9]) which indicate that FI parameters in 4D systems may exist, provided they are quantized in units of $2M_{pl}^2$ with $M_{pl}^2 = (8\pi G)^{-1}$ the reduced Planck mass squared, with G the 4D Newton's constant. Assuming this is possible, we study the characteristic p analog of supersymmetric vacua in the presence of an FI parameter. We also find that some notions of the resulting symplectic geometry carry over, giving us a notion of toric varieties in characteristic p. That being said, the fact that there is no notion of "big or small" in characteristic p means that there is little sense in which we can reach a semi-classical geometry in this case. We do, however, find that there is a suitable notion of quasi-locality, as specified by the Grothendieck topology. We note that the introduction of physical topoi has been discussed for example in references [10–15] but we leave to future work any attempt to align with the considerations found therein.

We believe that the present formulation also sheds light on some structures which appear in number theory, although we leave a full exploration for future work. For example, a well known quantity is the Hasse-Weil Zeta function [16, 17] for a variety V defined over a finite field $\mathbb{F}_q$ in characteristic p. A classic question in this subject is to compute the number of points as defined over a finite field $\mathbb{F}_{q^n}$. This is all packaged in terms of the Hasse-Weil Zeta function:

$$Z_{V,p}(z) = \exp\left(\sum_{n\geq 1} \#V(\mathbb{F}_{q^n})\frac{z^n}{n}\right). \tag{1.4}$$

Given the setup just explained, it is tempting to view this data as being specified by a supersymmetric index for a physical system [18], but now in characteristic p:

$$\sum_{n\geq 1} \mathrm{Tr}_n\left((-1)^{\mathbf{F}} z^n\right) = \log Z_{V,p}(z). \tag{1.5}$$

The appearance of an algebraic formulation for our physical system also enables us to analyze systems of direct relevance in string compactification. Typically, the string compactification geometry is treated as some large volume approximation to a more accurate quantum corrected system. Since, however, our entire formulation is algebraic, classic constructions such as geometric engineering [19–22] have characteristic p analogs, and allow us to formulate a conjectural correspondence between Calabi-Yau varieties of ADE singularities and certain topological field theories.

In the more general case where N is not necessarily a prime number, the arithmetic perspective also provides helpful hints for how to proceed. In this case, there are multiple prime factors, so it is fruitful to view our geometry as fibered over the "affine line with the origin deleted" $\mathrm{Spec}\,\mathbb{Z}^{\times}$, as spanned by the prime numbers. By performing a path integral over this bigger geometry, we can uniformly treat all primes at once. Viewed in this way, fixing a particular value of N amounts to a semi-classical approximation. Localization near a given prime divisor p of N then leads to a similar arithmetic interpretation for a general integer N.

Taking the large N limit also provides a route to recovering continuum notions such as more refined topological spaces. Taking this limit at the level of the action and fields actually produces, for $N = p^a$ with p a prime, a completion inside the p-adic numbers. If we instead take a limit as obtained on the "phase factors" of the path integral valued in $S^1 \subset \mathbb{C}^*$, we instead see a completion available in the real numbers. This provides a general route for recovering a continuum limit from our general discretized considerations.

Lastly, let us mention that the notion of looking for connections between number theory and physics is certainly not new to us. Indeed, many intriguing connections between p-adic numbers and strings have been appreciated for some time in both early work such as references [23–25] as well as in more recent work such as [26, 27]. The subject of p-adic strings also shows up in some approaches to studying tachyon condensation [28]. Connections between certain modular forms and arithmetic structures which appear in string theory have also been noted (see e.g. [29] for an example of this sort). It has even been appreciated for that the calculation of period integrals for some Calabi-Yau threefolds is amenable to techniques from number theory [30–32] (see also [33, 34]). Some proposals for quantum mechanics with different algebras include [35, 36]. We were also inspired by the arithmetic path integrals for certain topological field theories appearing in reference [37]. Perhaps closest in spirit to the present considerations is the work of reference [38] which sets up much of the necessary mathematical formalism in a setting close to the physical considerations explored here.

That being said, we have not tried to reconcile our present perspective with these considerations (with the notable exception of reference [38]) but it would be interesting to try. Indeed, our underlying motivation is somewhat different. First and foremost, our interest in the appearance of these structures is motivated by its potential use in studying physical, in principle experimentally accessible systems. That being said, we hope that this will lead to a two-way development, with physical notions helping to inform some questions in arithmetic

geometry, and conversely, that such number theoretic analogs of geometry can help in the search for a fundamental formulation of physics.

The rest of this note is organized as follows. We begin in section 2 by discussing the sense in which discretized systems allow for different choices of $\hbar$. In section 3 we begin with our first example, studying a discretized bosonic field. We generalize this construction in several ways, eventually arriving at a more geometric formulation amenable to study via methods in arithmetic geometry. Some aspects of the Hilbert space associated to these systems are discussed in section 4. Section 5 discusses the generalization to fermionic degrees of freedom, including a sketch of supersymmetric quantum mechanics in characteristic p. We also speculate on a physical interpretation of the Hasse-Weil Zeta function. In section 7 we present a proposal for geometric engineering in characteristic p. Section 8 discusses the extension of this analysis to more general number systems. We present our conclusions in section 9. In Appendix A we analyze some 1D quantum systems in characteristic p. In Appendix C we present an "alternative" supersymmetric quantum mechanics system in which we impose a different rule for Frobenius conjugation on Grassmann fields. Appendix D presents some evidence that FI parameters can be quantized in string constructions. Appendix E reviews some aspects of finite fields. In Appendix F we review the construction of inverse limits and in Appendix G we review some aspects of Witt vectors. Appendix H discusses some aspects of geometry in characteristic p.

2 Discretized Target Spaces

In this section we explore some of the consequences of discretizing a target space. In subsequent sections we shall refine this analysis, showing how additional structure can be maintained by working with geometry in characteristic p. It will prove convenient to summarize all of these limits as setting a convention in which fields take values on the integers and the reduced Planck constant is set to:

$$\hbar = \frac{N}{2\pi}. \tag{2.1}$$

We first motivate this discretization by way of a few examples. First, we consider a point particle, then a field on a 2D spacetime. We then make some brief philosophical comments. The aim of section 3 will be to formalize some of these features. Some additional details on 1D point particles in characteristic p are presented in Appendix A.

2.1 Point Particle

As a first example, suppose we have a particle moving in one spatial dimension. Classically, we can visualize this by introducing a function of time $Y(T)$, indicating the position of our particle. A common situation is an action of the form:

$$S[\Phi] = \int L dT = \int dT \ \left(\alpha m \left(\partial_T \Phi\right)^2 - \mathcal{V}(\Phi)\right), \tag{2.2}$$

where α is an integer and m is proportional to the mass of the particle. Note that we have not canonically normalized the fields. This will be important when we turn to the discretization of our system. Here, $\mathcal{V}(\Phi)$ denotes a potential energy and in what follows we shall assume that this is always taken to be a polynomial in the field Φ. In practice, one often expands a potential energy density about some background value of Φ, say Φ_0, and then analyzes the leading order terms of such an expansion. In this sense, we expect to get a "good approximation" by just dealing with polynomials of possibly very high degree. Indeed, one expects that in a theory of quantum gravity some higher order terms may actually capture strictly redundant information [2].

Quantum mechanically, we can use this as a starting point for the path integral. For example, we are instructed to sum over possible choices of functions $\Phi(T)$, each weighted by a factor $\exp(iS[\Phi]/\hbar)$. Correlation functions involving operators $\widehat{\mathcal{O}}$ built from the Φ's are obtained in the usual way by the formal relation (we leave time-ordering implicit):

$$\left\langle \Phi(T_{\text{final}}) \left| \widehat{\mathcal{O}}(T_m)...\widehat{\mathcal{O}}(T_1) \right| \Phi(T_{\text{init}}) \right\rangle \equiv \frac{\int_{\Phi_i}^{\Phi_f} [d\Phi] \ \exp(iS[\Phi]/\hbar)\, \mathcal{O}(T_m)...\mathcal{O}(T_1)}{\int_{\Phi_i}^{\Phi_f} [d\Phi] \ \exp(iS[\Phi]/\hbar)} \tag{2.3}$$

For brevity, in what follows, we shall leave the initial and final values of the field configurations implicit.

It is natural consider possible discretizations of the above system. For example, if we consider a particle which can only occupy points on a spatial lattice, there is a minimal spacing for values of the field. Doing so, however, introduces fresh complications. For example, if we simply posit $\Phi(T)$ takes values over the integers, then our notion of a time derivative ceases to make sense. There is a workaround which is available in systems where we are also only able to make measurements at discretized time steps. For example, an observer may be limited in how frequently they can actually measure the response of the system. In a Hamiltonian evolution of a given state such as:

$$|\Psi(T)\rangle = \exp(-i\widehat{H}T/\hbar)\,|\Psi(0)\rangle\,, \tag{2.4}$$

it may be that there is a minimal time resolution, so we can only ever access discretized time steps.

Discretization in the target space and the time direction suggests a way to proceed. First, we introduce a minimal step size for the field. Additionally, we introduce a minimal time step by which a particle can actually change. Changes in the energy are then also discretized. Making these changes amounts to the lattice approximation:

$$T \mapsto \tau_{\text{time}} t \tag{2.5}$$

$$\Phi(T) \mapsto \ell_{\text{target}} \phi(t) \tag{2.6}$$

$$\partial_T \Phi(T) \mapsto \frac{\phi(t+1) - \phi(t)}{\tau_{\text{time}}} \tag{2.7}$$

$$\int dT \mapsto \sum_t \tau_{\text{time}}, \tag{2.8}$$

$$\mathcal{V}(\phi) \mapsto \frac{m\ell_{\text{target}}^2}{\tau_{\text{time}}^2} V(\phi) \tag{2.9}$$

where we now assume $t, \phi(t), \phi(y) \in \mathbb{Z}$. Returning to the form of our action, we now have:

$$S[\phi] = \sum_t \frac{m\ell_{\text{target}}^2}{\tau_{\text{time}}} \left(\alpha \left(\phi(t+1) - \phi(t)\right)^2 - V(\phi)\right). \tag{2.10}$$

Evaluating correlation fuctions now proceeds just as in the ordinary path integral. For example, the integration over all paths is now replaced by discretized sums:

$$\int_{\Phi_i}^{\Phi_f} [d\Phi] \mapsto \sum_{\phi(t_f)} ... \sum_{\phi(t_i)} \delta(\phi(t_f) = \phi_f)\delta(\phi(t_i) = \phi_i) \tag{2.11}$$

where here, we have specified an initial and final field configuration. We have also dropped dimensionful factors associated with the path integral measure, since we have already now passed to the discretized setting.

The main thing we wish to explore is what happens when the dimensionless ratio involving these length scales and the Planck constant is held fixed:

$$\frac{1}{\hbar}\frac{m\ell_{\text{target}}^2}{\tau_{\text{time}}} = \frac{2\pi}{N}, \tag{2.12}$$

with N an integer. We could, of course, have jumped straight to this form of the phase factor in the path integral by working in natural units with all lengths and time steps set to one. In that case, we could assert:

$$\hbar = \frac{N}{2\pi}. \tag{2.13}$$

It is customary to work in natural units where $\hbar = c = 1$, but a priori we can consider more general choices, and they have no impact on the physics. Indeed, the classical limit is typically associated with the limit $\hbar \to 0$. Here, we are considering the opposite regime where all behavior is highly quantum, and so we have chosen to emphasize this by absorbing all these changes into the choice of the reduced Planck constant. In any event, the path integral is now weighted by factors of the form:

$$\exp(iS[\phi]/\hbar) = \exp\left(\frac{2\pi i}{N}\sum_t \left(\alpha\left(\phi(t+1)-\phi(t)\right)^2 - V(\phi)\right)\right). \tag{2.14}$$

This is where we encounter our first surprise. In these units, we observe that if all quantities in our system are discretized integers, then the only contributions we actually care about are obtained modulo N. Indeed, this is just because $\exp(2\pi i) = 1$.

One might also ask about observables in this sort of system. One class of operators which respect the observed mod N structure is given by "vertex operators" of the form:

$$U(t) = \exp(2\pi i\phi(t)/N). \tag{2.15}$$

Compared with our discussion of path integrals given above, the only difference is that now, we need not integrate over all paths, just their mod N residues. In the original continuum theory, these operators are mildly non-local, arising from expressions such as:

$$\exp\left(i\int_{t-\varepsilon}^{t+\varepsilon}\frac{dt'}{\tau_{\text{min}}}\frac{\Phi(t')}{\ell_{\text{min}}}\right), \tag{2.16}$$

for ε a small number. This amounts to a small amount of "smearing" in the original continuum theory.

Another comment has to do with the domain of the time coordinate. Assuming that the Hamiltonian has integer eigenvalues, we observe that the time evolution operator:

$$\exp\left(-\frac{i}{\hbar}\widehat{H}t\right) = \exp\left(-\frac{2\pi i}{N}\widehat{H}t\right) \tag{2.17}$$

repeats after at most N time steps.

2.2 2D Example

In the previous subsection we introduced a first example of a discretized system, observing the appearance of a natural mod N structure in the resulting path integral. We now present a generalization of this to the case of a 2D field theory. We focus on the case of a 2D non-linear sigma model of the sort one encounters in the study of string theory. In this case, the spacetime of the field theory consists of the worldsheet of the string. We focus on a lattice approximation to flat space $\mathbb{R}^{1,1}$ and consider the Polyakov action:

$$S[\Phi] = \frac{1}{4\pi\alpha'}\int d^2\sigma\ G_{AB}(\Phi)h^{ab}\partial_a\Phi^A\partial_b\Phi^B. \tag{2.18}$$

Here, α' has dimensions of length squared. This, of course, is the starting point for understanding perturbative strings moving in a target space with metric $G_{AB}(\Phi)$. It has been appreciated for some time that the minimal length scale in string theory is not set by $\sqrt{\alpha'}$, but can be far smaller, and involves the string coupling g_{string} as well [39, 40]. With this in mind, we explore the consequences of assuming that there is a minimal length scale which can be probed by our string. Much as in our discussion of the point particle, we first consider a rescaled version of the fields, writing:

$$\Phi \mapsto \ell_{\text{target}}\phi, \tag{2.19}$$

so that the ϕ's are valued in the integers. By the same token, we also replace all derivatives by lattice derivatives, with the worldsheet specified by points on the two-dimensional lattice $\mathbb{Z}\times\mathbb{Z}$. In this case, observe that since we are in two dimensions, the rescaling of the measure factor from the worldsheet integral cancels the rescaling of the lattice derivatives.

We would like to understand what happens when the dimensionless ratio involving the target space length scale and the string scale and the Planck constant is taken to be fixed as:

$$\frac{1}{\hbar}\frac{\ell_{\text{target}}^2}{4\pi\alpha'} = \frac{2\pi}{N}, \tag{2.20}$$

with N an integer. Much as in the case of the point particle example, we could have jumped straight to this form of the phase factor by working in natural units with all lengths and

time steps fixed to one and setting:

$$\hbar = \frac{N}{2\pi}. \tag{2.21}$$

In the context of string theory, taking the large N limit means the string tension passes to zero. This is clearly far away from the realm of classical geometry.

Pressing on, most of our discussion of this 2D example proceeds as in the 1D case. We can again also speak of vertex operators such as:

$$U(\sigma^a) = \exp(2\pi i k_A \phi^A(\sigma^a)/N). \tag{2.22}$$

Again, in the continuum theory this sort of expression comes about from a mildly non-local operator with some small amount of smearing, as per our discussion below equation (2.16). One can also entertain integer valued operators as well, and the prescription for calculating correlation functions is essentially the standard one for the path integral, just with a new domain of summation / integration.

2.3 Philosophical Comments

By now, the general procedure should be clear, at least for field theories specified by scalars. We discretize the target space, and also the spacetime of the field theory. The case of two dimensions is a bit special in this regard, because the actual lattice spacing of the spacetime drops out from our expression for $\hbar$. In more general systems with non-trivial operator scaling dimensions, similar considerations would likely also apply. This motivates us to study systems in which the values of fields are restricted to integers, with the reduced Planck constant set to the value:

$$\hbar = \frac{N}{2\pi}. \tag{2.23}$$

At a conceptual level, introducing this sort of discretization is appealing for a number of reasons. As we have already mentioned, there is a sense in which any measurement by an observer already comes in "quantized units." Indeed, there is a strict difference between the real numbers and those which are actually computable. An additional comment is that fundamental physics makes reference to quantities such as a Planck time, Planck length, and Planck energy scale. All of these signal some basic intuition that there is a minimal unit of measurement. Note also that the class of operators which naturally enter in this setting include some mild amount of non-locality, such as:

$$\exp\left(i \int_{\varepsilon_D} \frac{d^D x}{\ell_{\text{min}}^D} \frac{\Phi(x)}{\Lambda^\Delta}\right) \tag{2.24}$$

where ℓ_{min} refers to a minimal length scale, and Λ is a mass scale, and Δ is the engineering dimension for a field Φ. Here, the integral takes place over a small region ε_D, indicating a

mild amount of smearing / averaging. This is also in line with the expectation that there are limits to statistical inference in quantum gravity [41].

We also remark that in some cases, this intuition has recently been sharpened in the context of the Swampland conjectures (see e.g. [1, 42, 43] and references [44, 45] for reviews). One of the recurring themes in this line of research is to explore the impact of the Planck scale on long distance physics, particularly low energy effective field theories. Naive extrapolation of an effective field theory is expected to produce various pathologies, and one potential way around this is to discretize various physical structures (see e.g. [46]).

On the other hand, there are also well-known drawbacks to discretization. For one, introducing an explicit lattice cutoff immediately destroys Lorentz invariance of the system. In lattice field theory, it is common to fine-tune all parameters so that Lorentz invariance is recovered at long distances. Such an option may not be available here since we are also discretizing the parameters of the system. Another difficulty is that the proper treatment of fermions, let alone supersymmetry is rife with technical (though not insurmountable) difficulties. See for example, [47] for some recent discussion on these points. Along these lines, any notion of quantum gravity on a lattice is again potentially quite problematic since the lattice itself would need to fluctuate. An additional concern is that one of the powerful probes of quantum locality comes from analyticity of the S-matrix. Much of the power of results in scattering amplitudes comes from the fact that the S-matrix can be analytically continued to "unobservable" large and complex values of momenta. Sacrificing this in the name of discretization is also potentially quite problematic.

We propose to balance these competing considerations using the geometry of numbers.[2]

[2]In the poetic sense, not the strict sense of Minkowski.

3 Discretization in Characteristic p

In the previous section we presented an intriguing observation that some discretized systems have close contact with some crude features of arithmetic modulo N. We also saw, however, that a direct lattice approximation produces some potentially unpleasant features, particularly if we wish to maintain contact with analytic structures which appear so central to many aspects of fundamental physics. The main idea we develop here will be to consider an alternative interpretation of such discretized systems in which we leverage the "analytic structure" present in arithmetic geometry, namely algebraic geometry in characteristic $p > 0$.

With this in mind, in this section we confine our attention to the special case where $N = p$ is an odd prime number. Much of what we develop also works (with suitable amendments) in characteristic 2, but we do not discuss this special case in what follows. The main issue we need to develop is a suitable notion of a path integral, as defined by an action principle. This will require us to provide a notion of:

$$\int_{\Phi_i}^{\Phi_f} [d\phi] \ \exp(iS[\phi]/\hbar) \equiv \sum_{\substack{\text{smth } \phi: X \to Y \\ \phi(t_f)=\Phi_f \\ \phi(t_i)=\Phi_i}} \exp\left(\frac{2\pi i}{p} S[\phi]\right), \tag{3.1}$$

as well as insertions of operators, as in our discussion around equation (2.3). We interpret this in the following subsections in increasing levels of abstraction, but the main idea will be to view it as a sum over all possible morphisms $\phi : X \to Y$.[3] We will also need a notion of a Lagrangian density $\mathcal{L}[\phi]$, which, for a fixed ϕ, is locally just a polynomial over a finite field. The action is given by evaluating $\mathcal{L}[\phi]$ at all the points of X and summing up. Or alternatively, we can just evaluate S at all the points and take the product:

$$\prod_{x \in X} \exp\left(\frac{2\pi i}{p} S_x\right). \tag{3.2}$$

We will impose a notion of "unitarity" by which we mean that the complex phases all have norm one. We enforce this through the condition that the evaluation of the action in this way produces a quantity valued in $\mathbb{F}_p$. Finally, the "limits of integration" Φ_i and Φ_f indicate fixed values of our morphism at marked locations on X specified by the divisors $t_i = 0$ and $t_f = 0$ We return to the quantum interpretation in section 4.

The rest of this section is organized as follows. We begin by developing our analysis over the field $\mathbb{F}_p$. We follow this with a discussion of finite field extensions such as $\mathbb{F}_q$, and finally the algebraic closure $\overline{\mathbb{F}_p}$. We then extend this to varieties over finite fields in characteristic p.

[3] Here, our main requirement is that we can present our map in terms of a polynomial in local coordinates.

3.1 Physics on $\mathbb{F}_p$

We begin by revisiting our discretized bosonic system, but now with an eye towards maintaining additional analytic structure. We do this so that we can keep additional symmetries manifest, and also so that we can eventually generalize to systems with other sorts of degrees of freedom (such as fermions, vector bosons and gravitons).

Let us return, then, to nearly the beginning. We now posit that we are working with a quantum system with integer values for our fields and in which the reduced Planck constant is discretized in units of p:

$$\hbar = \frac{p}{2\pi}. \tag{3.3}$$

We also assume that all observables of interest are really specified modulo p. For example, we assume the kinetic and potential energies of the action take values in the integers, and that the physical operators of interest are all specified by fields modulo p. Some important examples to keep in mind include exponentiated fields which take values in the character group for $(\mathbb{F}_p, +)$, viewed as an additive group.

Working over the integers modulo p, we arrive at a finite number field, $\mathbb{F}_p$. We review some properties of finite number fields in Appendix E, and we refer the interested reader there for a brief discussion of this rich subject. Compared with more familiar fields such as the rational numbers, real numbers or complex numbers (or even the p-adics), adding up any element $\phi \in \mathbb{F}_p$ by a multiple of p results in zero, namely $p\phi = 0$. A field which satisfies this property is said to be in characteristic p. If this property does not hold, we say that the field is in characteristic zero.

One consequence of this is that there is no natural notion of a metric we can provide, though as we explain, there is a close characteristic p analog which retains some of the structure one would want of a physical metric. That being said, many analytic structures of geometry do remain intact provided we are flexible in our notion of what counts as a physical morphism.

Having specified that our physical fields actually take values in a finite field, we could in principle just repeat our lattice construction now, by specifying for each point $x \in X_{\text{spacetime}}$ on the spacetime lattice a value $\phi(x) \in \mathbb{F}_p$. So, we can view $X_{\text{spacetime}} \simeq \mathbb{Z}^D$ as a D-dimensional lattice with one direction singled out for time. We will shortly generalize this to move away from this limited choice.

As we have already mentioned, the notion of a finite derivative is a bit awkward, especially when there is no natural notion of "metric." To develop a suitable replacement, we will first consider a natural class of objects given by polynomials in some number of variables, written as $\mathbb{F}_p[t_1, ..., t_D]$. Our physical field ϕ can now be viewed as a polynomial in these variables:

$$\phi(t_1, ..., t_D) = \sum_{i_1,...,i_D} \phi_{i_1...i_D} (t_1)^{i_1} ... (t_D)^{i_D}, \tag{3.4}$$

where each of the $\phi_{i_1 \cdots i_D}$ is an element of $\mathbb{F}_p$. Taking a derivative proceeds just as in ordinary calculus. Note that when the exponent is a multiple of p, this derivative is automatically zero, a consequence of working in characteristic p. In principle, one can just continue to take ordinary derivatives, but a slightly more sophisticated option is to consider a Hasse derivative.[4] With all of these considerations in mind, we see that rather than dealing with a finite difference, it is in some sense simpler to work with derivatives of polynomials. Of course, once we evaluate the derivative we just compute the polynomial at the prescribed (integral) spacetime point, reduced modulo p. As an additional comment, we note that the space of polynomials is of course infinite. To generate concrete approximations we can always truncate to a fixed degree. This can then be used to match up with the lattice approximation.

Defining actions of relevance for physical systems is now straightforward. We illustrate by way of example. Given a polynomial $\phi \in \mathbb{F}_p[t_1, ..., t_D]$, we introduce a Lagrangian density $\mathcal{L}[\phi]$ as a functional on a given choice of ϕ. By composition of maps, this can also specifies an element of $\mathbb{F}_p[t_1, ..., t_D]$. As a specific example, we take:

$$\mathcal{L}[\phi] = \alpha \left((\partial_1 \phi)^2 - (\partial_2 \phi)^2 - ... - (\partial_D \phi)^2 \right) - V(\phi), \tag{3.5}$$

where $\alpha \in \mathbb{F}_p$ and $V \in \mathbb{F}_p[\phi]$. To extract a number, we now sum over the points in the spacetime using the evaluation map:

$$S[\phi] = \sum_{(x_1, \cdots, x_D) \in X_{\text{spacetime}}} \mathcal{L}(t_1 = x_1, ..., t_D = x_D), \tag{3.6}$$

where now we simply treat $\mathcal{L}$ as a polynomial in the formal parameters of $\mathbb{F}_p[t_1, ..., t_D]$, and then evaluate. In this case, the phase factor of the path integral defines a character map on the additive group of the finite field:

$$\exp : (\mathbb{F}_p, +) \to U(1) \tag{3.7}$$

$$S \mapsto \exp\left(\frac{2\pi i}{p} S \right). \tag{3.8}$$

As it stands, we are summing over a lattice with an infinite point set. This means in particular that S evaluated on this physical field configuration may not be well-behaved. On the other hand, since $\phi^p = \phi$ in $\mathbb{F}_p$, we are typically summing over "multiple copies" of the same spacetime point when we evaluate over all the integers.

One possibility is to just reduce the lattice $\mathbb{Z}^D$ modulo p, so that we instead deal with

[4] In a variable t, the rth Hasse derivative is defined via its action on a monomial t^n: $\mathcal{D}^{(r)} t^n = \frac{n!}{n!(n-k)!} t^{n-k}$ when $0 \leq r \leq n$, and otherwise vanishes. The advantage of using this derivative is that it allows more terms to remain non-zero. Another helpful feature is that an analog of Taylor's theorem holds in terms of a local parameter x of a variety X: $f = \sum_r \mathcal{D}^{(r)}(f) \cdot x^r$.

D-dimensional affine space in characteristic p:

$$X_{\text{spacetime}} = \underbrace{\mathbb{F}_p \times ... \times \mathbb{F}_p}_{D \text{ times}} = \mathbb{A}^D. \tag{3.9}$$

In the spirit of algebraic geometry, we can also consider more general algebraic varieties in characteristic p. These more general choices can have more or less points depending on the choices of hypersurface equations. See also Appendix H.

With this in mind, we can already anticipate that it will be fruitful to expand our horizons, allowing X to be specified as the zero set of more general polynomials in characteristic p. By a similar token, we can also enlarge the target space Y in a similar way. In all these cases, there is a suitable generalization of a polynomial to maps of the form:

$$\phi : X_{\text{spacetime}} \to Y_{\text{target}}, \tag{3.10}$$

where so far, we have restricted to affine spaces. The main idea is to view these ϕ's as locally specified by polynomials, and to then construct a Lagrangian from these fields.

Let us now turn to a few of the important distinctions from quantum field theory in characteristic zero. First of all, we observe that for any element $\phi \in \mathbb{F}_p$, we have the identity:

$$\phi^p - \phi = 0. \tag{3.11}$$

If we evaluate the physical potential $V(\phi)$ at a given point of $X_{\text{spacetime}}$, then high degree terms in this polynomial in ϕ are in some sense redundant. All of the physical information is already contained in the expansion:

$$V(\phi) = V_0 + V_1\phi + ... + V_{p-1}\phi^{p-1}, \tag{3.12}$$

so provided we only evaluate on $X_{\text{spacetime}}$, without loss of generality we can simply work solely in terms of this finite set of coefficients rather than the infinite set which is customary in effective field theory. Similar considerations hold for the kinetic term and higher derivative terms. One consequence of this is that there is indeed a redundancy in higher order coefficients. This is in line with expectations on the constraints of a low energy effective field theory indicated in various Swampland conjectures (see e.g. [2, 48]). One can, of course, always include such higher degree terms, and this is useful in coming up with a better approximation scheme. It is, however, strictly speaking redundant information.

Another surprising feature of our Lagrangian is that the kinetic term of our scalar field theory seems to make reference to a Lorentzian signature metric. Of course, this is an illusion; in characteristic p we also have:

$$(\partial_1\phi)^2 - (\partial_2\phi)^2 - ... - (\partial_D\phi)^2 = (\partial_1\phi)^2 + (p-1)((\partial_2\phi)^2 + ... + (\partial_D\phi)^2), \tag{3.13}$$

so distinctions between Euclidean and Lorentzian signature spacetimes also fall by the wayside.

3.2 Physics on $\mathbb{F}_q$

Our discussion thus far has focussed on varieties defined over $\mathbb{F}_p$, the integers modulo p. We now extend these considerations to other finite fields such as $\mathbb{F}_q$. Recall from Appendix E that every finite field in characteristic p has $q = p^n$ elements for some $n \geq 1$. The field $\mathbb{F}_q$ can be constructed as the splitting field of an irreducible degree n polynomial over $\mathbb{F}_p$. We can think of this field as obtained by adjoining a single root α of such a polynomial, writing $\mathbb{F}_q = \mathbb{F}_p(\alpha)$. From the perspective of Galois theory, we can view $\mathbb{F}_q$ as a vector space over the field $\mathbb{F}_p$. A convenient basis of vectors is given by the p^{th} powers of this root, so we can represent any element in $\mathbb{F}_q$ as a power series of the form:

$$\phi = \sum_{j=1}^{n} \phi_j \alpha^{p^{j-1}}. \tag{3.14}$$

Indeed, the Frobenius automorphism:

$$F : \mathbb{F}_q \to \mathbb{F}_q \tag{3.15}$$

$$\phi \mapsto \phi^p \tag{3.16}$$

simply pemutes these powers. Here, we have used the fact that in characteristic p, $(a+b)^p = a^p + b^p$.

The appearance of an n-dimensional vector space over $\mathbb{F}_p$ has a clear interpretation in terms of the physical degrees of freedom we have already introduced. Instead of considering a single particle moving in a 1D target spanned by the integers, we can consider n such particles. We can denote this by an n-component vector with coordinates $\phi^{(1)}, ..., \phi^{(n)}$. This also makes it clear that we can construct corresponding Lagrangians involving our n particles. As an example, we can construct a kinetic term for our n particles given by:

$$G_{AB} \left(\partial_1 \phi^A \partial_1 \phi^B - \partial_2 \phi^A \partial_2 \phi^B - ... - \partial_D \phi^A \partial_D \phi^B \right), \tag{3.17}$$

where repeated indices are summed over. Here, we have introduced a symmetric bilinear form with entries G_{AB}:

$$G : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p, \tag{3.18}$$

which specifies a "dot product" for the system.

Now, instead of working in terms of these n-component vectors, we could alternatively view this as a single particle on a one-dimensional target $\mathbb{F}_q$. In characteristic zero, we implicitly do exactly this sort of thing when we view a complex scalar field as a linear

combination of two real scalars. In contrast to the complex numbers, however, there are many analogs of the imaginary numbers which we can adjoin to $\mathbb{F}_p$. As explained in Appendix E, we can alternatively view the bilinear forms of line (3.18) as an $\mathbb{F}_p$ valued pairing:

$$G : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p. \tag{3.19}$$

So, it is a matter of taste whether we wish to work in terms of many physical fields, or in terms of a single $\mathbb{F}_q$ valued field. The main condition we need to enforce in this generalized perspective is that our action takes values in $\mathbb{F}_p$ rather than the larger field $\mathbb{F}_q$. The reason is much the same as in other quantum systems: We need to ensure that there is a proper notion of unitary time evolution, and this would be destroyed if our action ended up being valued outside the integers modulo p.

One way to construct $\mathbb{F}_p$ valued actions is to demand that all evaluations are invariant under the Frobenius automorphism. Indeed, $\mathbb{F}_p$ is the only subfield of $\mathbb{F}_q$ invariant under this automorphism. Given an element $\phi \in \mathbb{F}_q$, common invariants include the Trace and Norm:

$$\text{Trace}(\phi) = \sum_{i=0}^{n-1} F^i(\phi) \tag{3.20}$$

$$\text{Norm}(\phi) = \prod_{i=0}^{n-1} F^i(\phi). \tag{3.21}$$

The Trace is clearly useful in producing invariant kinetic terms, while both the Trace and Norm are useful in constructing invariant potential energy densities. At a more general level, our only true demand is that our action have "local" interaction terms. There are, however, physically motivated choices, as we have indicated above.

Having seen that we can extend the target space to be $\mathbb{F}_q$, one might ask whether a similar extension to spacetimes defined over $\mathbb{F}_q$ is well-motivated. Of course, at a formal level, nothing stops us from doing so. Indeed, so long as our action continues to evaluate to elements in $\mathbb{F}_p$, there is no reason not to make this extension. From a physical perspective, one can view this as supplementing our original discretized spacetime by additional points. A perhaps more satisfying answer is that the target space for a string is interpreted as another spacetime in its own right. In this context, then, it is again sensible to allow for such field profiles.

3.3 Physics on $\overline{\mathbb{F}_p}$

Proceeding in this way, we can now ask about the interpretation of taking the target space to be $\overline{\mathbb{F}_p}$, the algebraic closure of our finite field. One might view this as playing the analogous role to that which the complex numbers play in relation to the real number numbers. Of course, here, there are many more analogs of the "imaginary numbers" available!

One important remark is that the algebraic closure has infinite order. This means that the procedure for computing values of the action used previously will not really work, since the "evaluation map" procedure requires us to sum over all the points of a variety. Now, in characteristic zero we could introduce a measure on our spacetime and use this to suitably integrate over the Lagrangian density.

What can we do in the present case? The main idea we use to define the path integral phase factor $\exp(iS/\hbar)$ in this case is to observe that actually, our evaluation can in instead be viewed as a product over characters. Recall that for a field k, the group of additive characters involves an "exponential map" to $U(1) \subset \mathbb{C}^\times$. Phrased in this way, we can, for each point in a finite field first compute the additive character, and only then take the product. In the obvious notation, the evaluation of the phase factor for the path integral can instead be written as:

$$\prod_{x \in X} \exp\left(\frac{2\pi i}{p} S_x\right) \in U(1) \subset \mathbb{C}^\times. \tag{3.22}$$

The advantage of setting things up this way is that now, we can consider a sequence of containments:

$$k_0 \subset k_1 \subset k_2 \subset ... \subset k_m \subset ... \subset \overline{\mathbb{F}_p}, \tag{3.23}$$

and with it the corresponding sequence of characters obtained from evaluation on a given field configuration:

$$\chi_{k_0}, \chi_{k_1}, ..., \chi_{k_m}, ... \tag{3.24}$$

Of course, there is no guarantee that such a sequence will converge in the metric topology of $\mathbb{C}$. Additionally, there is of course more than one way to build a nested containment of finite field extensions contained in $\overline{\mathbb{F}_p}$. To have a well-defined limit, we require that any such sequence converges to the same point in $U(1)$, and when it does, we write the limit as:

$$\lim_{\longrightarrow} \chi_{k_n} \equiv \chi_{\overline{\mathbb{F}_p}}. \tag{3.25}$$

At a practical level, however, we can simply truncate a given sequence. Proceeding in this way, we can speak of path integrals over algebraically closed fields such as $\overline{\mathbb{F}_p}$.

In this enlarged setting we can also contemplate the physical meaning of the Frobenius automorphism, namely the generator of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$. For finite fields $\mathbb{F}_q$ we interpreted the Frobenius map as permuting a collection of $\mathbb{F}_p$ valued fields, and the same considerations apply here as well, albeit for a now infinite collection of particles. This provides another way for us to interpret our path integral phase factor of line (3.22) as the many body phase factor of a formally infinite collection of particles.

3.4 Physics on Varieties in Characteristic p

Our discussion so far has mainly focused on the simplest examples of spacetimes and target spaces. We can also consider more general geometries by specifying varieties in characteristic p. The procedure for constructing such spaces is a standard one from algebraic geometry, and it carries over essentially unchanged. We construct affine patches of a variety by specifying the zero set for some polynomials. Then, we glue these patches together to produce a our more general variety. We now speak of our physical fields as specified by rational maps of the form:

$$\phi : X_{\text{spacetime}} \dashrightarrow Y_{\text{target}}. \tag{3.26}$$

In terms of the local coordinate rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ for $x \in X$ and $y \in Y$, this means that we will allow our physical fields y to be written as ratios:

$$\phi = \frac{P}{Q}. \tag{3.27}$$

The reason we should allow such maps is that in most geometries of interest, working with just polynomials will not produce enough "interesting" maps. This is the point of allowing birational maps. The price we pay in doing this is that we inevitably encounter possible singularities in the evaluation of our action. This is actually not that problematic, it just indicates the physical presence of a source, and means that we need to specify some choice of boundary conditions in the path integral with prescribed pole structure for field configurations. This is often referred to as inserting a defect operator in the path integral. This is also customary in specifying asymptotic scattering states. For all these reasons, we shall remain flexible in our notion of a physical field. The proper notion of the path integral would seem to involve integrating over rational morphisms $X_{\text{spacetime}} \dashrightarrow Y_{\text{target}}$.

In this more general setting, we can now also provide a more geometric formulation for the terms appearing in our action. Consider, for example, the kinetic term of a bosonic field theory. In characteristic p, we can still speak of the cotangent space to a point, so we can consider the pullback map on the cotangent spaces:

$$\phi^* : T^*_y Y \dashrightarrow T^*_x X, \tag{3.28}$$

a differential such as $d\phi$ has the standard local form:

$$d\phi^A = \frac{\partial \phi^A}{\partial x^a} dx^a. \tag{3.29}$$

Indeed, as we have repeatedly emphasized, much of the algebro-geometric structure typically used in characteristic zero carries over to characteristic p (with suitable amendments).

Additionally, we can introduce symmetric bilinear forms:[5]

$$G : T^*Y \otimes T^*Y \to \mathbb{F}_q \to \mathbb{F}_p \tag{3.30}$$

$$h : T^*X \otimes T^*X \to \mathbb{F}_q \to \mathbb{F}_p, \tag{3.31}$$

where we have factored this map through the Trace map. We refer the reader to Appendix H for the definition of the cotangent space in characteristic p. The main point is that even though there is little notion of "distance," in these spaces, we can still introduce symmetric bilinear forms valued on the "observable" numbers.

Consequently, we can now specify far more general actions in characteristic p as well. Superficially, there is little change from our earlier considerations. For example, a non-linear sigma model metric on a spacetime X can be written as:

$$S = \sum_{x \in X} \sqrt{\det h} h^{ab} G_{AB} \partial_a \phi^A \partial_b \phi^B. \tag{3.32}$$

Here, each of the ϕ's is to be interpreted as a rational map from $X \dashrightarrow Y$, and derivatives of local coordinates on X are computed as before. Again, our only demand at this point is that for each such physical field configuration, the action remains valued in $\mathbb{F}_p$, this being the analog of unitarity in characteristic p.

Observe also that the expression we have arrived at is naturally covariant, even though we are working on a discretized spacetime and target space. Indeed, under a non-singular (up to a lower codimension space) change of coordinates, the standard rules of tensor calculus hold. In characteristic p, the analog of a local analytic isomorphism (i.e. a diffeomorphism) is an étale morphism (a special case of a smooth morphism in which the relative dimension is zero). The main thing we want to ensure is that we have the characteristic p analog of the inverse function theorem for manifolds in characteristic zero. Demanding that the Jacobian is invertible ensures this. See Appendix H for an extremely brief discussion of étale morphisms.

As a final generalization, now that we have moved to a far more geometric language, it is natural to ask whether we can start to incorporate some additional sorts of degrees of freedom, such as vector bosons and even gravitons. A priori, there does not appear to be any issue with doing this in characteristic p.

In fact, some of the mathematical formulation of vector bosons and gravitons in characteristic p has been carried out in reference [38]. The key feature for us is that using a suitable notion of localization of sheaves, the resulting formulae for the gauge connection and "metric" behave completely analogously to what one has in the characteristic zero case!

For example, for an abelian gauge field V_a, we can consider gauge transformations such

[5]Here we use the physicist convention for a metric, $ds^2 = G_{AB} dY^A dY^B$ so we evaluate on the cotangent bundle rather than the tangent bundle.

as:

$$V_a \mapsto V_a + \partial_a \varepsilon, \tag{3.33}$$

where ε is to be interpreted locally as a polynomial in the coordinate ring of the variety. Note that the field strength:

$$F_{ab} = \partial_a V_b - \partial_b V_a \tag{3.34}$$

is invariant under such gauge transformations, independent of the characteristic. Quantities such as $F_{ab}F^{ab}$ can then be used to build gauge invariant actions in the standard way. Here, we raised and lowered indices with a symmetric bilinear form h^{ab}.

Constructing a scalar degree of freedom charged under such a field is also straightforward. For example, given α, we can impose the condition for gauge transformations:

$$\alpha \mapsto \alpha - \varepsilon, \tag{3.35}$$

so the quantity:

$$\frac{1}{2} g^{ab}(\partial_a \alpha + V_a)(\partial_b \alpha + V_b) \tag{3.36}$$

is also gauge invariant.

We now provide a more systematic treatment of gauge interactions, but still focus primarily on motivated examples. From the outset, one complication we face is that in characteristic zero we can easily pass from elements of a Lie algebra to a local presentation of an element in the Lie group via the exponential map. In characteristic p we do not have this luxury, so caution is warranted.

We begin by constructing a field theory with $SO(n, \mathbb{F}_p)$ gauge interactions. Consider the theory of an n-component vector of $\mathbb{F}_p$ valued scalar fields which we denote as ϕ^A. We introduce a fixed symmetric bilinear form $G_{AB} = \delta_{AB}$. We can then consider Lagrangians such as:

$$L = \frac{1}{2} G_{AB} \partial_a \phi^A \partial^a \phi^B - \lambda \left(G_{AB} \phi^A \phi^B - \xi \right)^2, \tag{3.37}$$

with λ and ξ fixed parameters. We observe that this Lagrangian enjoys an $SO(n, \mathbb{F}_p)$ symmetry. By this, we mean the set of $n \times n$ matrices with entries in $\mathbb{F}_p$ such that $M^T M = \mathbb{I}_{n \times n}$, the identity.

We now attempt to gauge this global symmetry. In characteristic zero, we would introduce local gauge transformations, as designated by g_x, so that for each point $x \in X_{\text{spacetime}}$, we get an element in the symmetry group. We would like to attempt something similar in characteristic p. The first complication we encounter is that all our fields are being represented as polynomials, so one might rightly ask whether this can be extended to the present setting. Indeed, the proper framework for carrying this out is to consider a sheaf $\mathcal{V}$ such that each stalk $\mathcal{V}_x$ admits a group action by $SO(n, \mathbb{F}_p)$. Then, we can speak of the condition $g_x^T g_x = \mathbb{I}_{n \times n}$. To get this into a more practical form recognizable to a physicist, we can also consider the space of $n \times n$ matrices with entries in $\mathbb{F}_p(t)$, the field obtained by adjoining

the formal element t. Then, the condition $g(t)^T g(t) = \mathbb{I}_{n\times n}$ specifies a set of $n \times n$ matrices with entries in $\mathbb{F}_p(t)$ which satisfy the desired gauge transformation properties. With this in place, we can now introduce a vector potential V_a. Near a point $x \in X$, each component of this vector is to be viewed as an element of $\mathfrak{so}(n, \mathbb{F}_p) \otimes \mathcal{O}_{X,x}$, namely we impose the condition $(V_a)^T = -V_a$ on our local polynomial expressions. Globally, of course, we should think of $\partial_a + V_a$ as specifying a connection on our sheaf. We wish to consider gauge transformations of the form:

$$V_a \mapsto g^{-1} V_a g + g^{-1} \partial_a g, \tag{3.38}$$

where each g is interpreted as above. The important point for us is that even though we do not have the exponential map, we can still consider the group action of $SO(n, \mathbb{F}_p)$ on $\mathfrak{so}(n, \mathbb{F}_p)$.

At this point, the discussion is so close to that of characteristic zero that we can simply write down the standard action obtained from minimal coupling:

$$L = \frac{1}{2} G_{AB} (\partial_a \phi^A + (V_a)^A{}_{A'} \phi^{A'})(\partial^a \phi^B + (V^a)^B{}_{B'} \phi^{B'}) - \lambda \left(G_{AB} \phi^A \phi^B - \xi \right)^2 . \tag{3.39}$$

We can also extend this to other characteristic p fields such as $\mathbb{F}_q$, as per our discussion in earlier sections.

A pleasant feature of the group $SO(n, \mathbb{F}_p)$ is that all entries are already valued in $\mathbb{F}_p$, so the characteristic p analog of unitarity is guaranteed. What about other gauge groups? Perhaps the most familiar in physics applications characteristic is the group $U(n)$. To get something like that in the present context, we need to have a suitable notion of hermitian conjugation, as well as a suitable notion of an "imaginary number."

The appropriate notion of complex conjugation in characteristic p is Frobenius conjugation. Working over a base field $\mathbb{F}_q$, there is a notion of Frobenius conjugation given by $F_q(\phi) = \phi^q$ which holds fixed all elements of $\mathbb{F}_q$. We can then consider a quadratic extension by an element $\widehat{i}_q$ specified by the condition:

$$F_q(\widehat{i}_q) = -\widehat{i}_q, \tag{3.40}$$

which acts as the characteristic p analog of complex conjugation. We remark that this element may not square to -1. For example, in $\mathbb{F}_5$, observe that $3^2 = -1$ but that $3 \in \mathbb{F}_5$ whereas $\widehat{i}_5$ is not (since it is not invariant under Frobenius conjugation). To proceed more systematically, we will instead seek out a root of the polynomial equation:

$$x^q = -x, \tag{3.41}$$

and we denote one such root by $\widehat{i}_q$. Observe that by design, we have:

$$F(\widehat{i}_q) = (\widehat{i}_q)^q = -\widehat{i}_q. \tag{3.42}$$

Since $\widehat{i}_q$ is not invariant under Frobenius conjugation, it is not an element of $\mathbb{F}_q$. Note, however, that its square $(\widehat{i}_q)^2$ is invariant, and is therefore an element of $\mathbb{F}_q$.

We can now introduce an analog of hermitian conjugation as follows. Given an $n \times n$ matrix M with entries in $\mathbb{F}_q(\widehat{i}_q)$, write:

$$H = H_1 + \widehat{i}_q H_2, \tag{3.43}$$

with H_1 and H_2 some $n \times n$ matrices with entries in $\mathbb{F}_q$. We define a daggering operation:

$$H^\dagger \equiv H_1^T - \widehat{i}_q H_2^T. \tag{3.44}$$

The group of unitary matrices is now defined by writing:

$$U(n, \mathbb{F}_q(\widehat{i}_q)) = \left\{ H \in GL\left(n, \mathbb{F}_q(\widehat{i}_q)\right) | H^\dagger H = \mathbb{I}_{n\times n} \right\}. \tag{3.45}$$

Now, in characteristic zero, we could start with the theory that enjoys an $SO(2n)$ global symmetry and consider gauging a subgroup such as $U(n)$. In characteristic p, this is a bit more subtle because the notion of hermitian conjugation now makes reference to a specific notion of $\widehat{i}_q$.[6]

We now build a Lagrangian which enjoys the global symmetry $U(n, \mathbb{F}_p(\widehat{i}_p))$. Consider a theory of $2n$ $\mathbb{F}_p$-valued scalars $\phi^1, ..., \phi^{2n}$. We construct the complexified combinations:

$$\varphi^A = \phi^A + \widehat{i}_q \phi^{A+n} \tag{3.48}$$

$$\overline{\varphi}^{\overline{A}} = \phi^A - \widehat{i}_q \phi^{A+n}. \tag{3.49}$$

In this case, we can introduce a suitable bilinear pairing $G_{\overline{A}B}$ and write:

$$L = G_{A\overline{B}}(\partial_a \varphi^A + (V_a)^A{}_{A'} \varphi^{A'})(\partial^a \overline{\varphi}^{\overline{B}} - \left(\overline{V}^a\right)^B{}_{B'} \overline{\varphi}^{\overline{B'}}) - \lambda \left(G_{AB}\phi^A \phi^B - \xi\right)^2. \tag{3.50}$$

Here, we have introduced a vector potential V_a. Near a point $x \in X$, each component of V_a is to be viewed as an element of $\mathfrak{u}(n, \mathbb{F}_p(\widehat{i}_p)) \otimes \mathcal{O}_{X,x}$, namely we impose the condition $(V_a)^\dagger = -V_a$ on our local polynomial expressions. Globally, of course, we should think of

[6] In some cases, however, it is possible to proceed in a similar fashion to characteristic zero. To illustrate, suppose that we fix our base field to be $\mathbb{F}_p$. We then make the further assumption that $\widehat{i}_p^2 = -1$. Note that this restricts us to $p = 3 \bmod 4$. To see why, write $p = 3 + 4n$. Next, observe that

$$-\widehat{i}_p = \left(\widehat{i}_p\right)^p = \left(\widehat{i}_p\right)^{3+4n} = \left(\widehat{i}_p\right)^3, \tag{3.46}$$

which implies:

$$-1 = \left(\widehat{i}_p\right)^2. \tag{3.47}$$

In this case, we can use the standard manipulations used in characteristic zero. This, however, imposes restrictions on the prime p which we would like to avoid.

$\partial_a + V_a$ as specifying a connection on our sheaf.

Turning next to the analog of the graviton, we have also mentioned that there is really no issue in defining a symmetric bilinear form defined over each point of a scheme X. We mainly need to impose a local equivalence relation:

$$h_{ab} \sim h_{ab} + \partial_a \xi_b + \partial_b \xi_a, \tag{3.51}$$

for $\xi \in T^*X$. Summing over all equivalence classes in this way provides the path integral instruction for how to sum over the space of such "metrics."

As a final amusing comment, note that the definition of standard Riemannian geometry tensors only makes algebraic reference to the quantity h_{ab} and its derivatives. This would seem to suggest that we can even borrow the standard actions for gravity. As we have already noted, however, higher derivative terms in any effective action will eventually truncate in working over a finite field. We leave a full treatment of these interesting issues for future work.

4 Hilbert Space Considerations

Up to now, our main emphasis has been on developing a path integral formalism for physical theories in characteristic p. To a certain extent, this provides an operational definition of our physical theory, because we can use this framework to compute correlation functions of operators. Of course, what this leaves implicit is the actual structure of the Hilbert space in question. Our aim in this subsection will be to explain how this additional physical structure comes about in characteristic p.

To begin, we need to assume a notion of time in our characteristic p spacetime. Assuming we have specified a global section of $T^*X \otimes T^*$, we can, at least locally, specify a time coordinate, and in what follows we assume this can be done more globally. In fact, for ease of exposition, we will assume that our space X factorizes as $X_{\text{time}} \times X_{\text{space}}$, where we view X_{time} as the affine line or its projectivization, and X_{space} as the spatial directions. This can be generalized to fibrations of the form:

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ & & \big\downarrow \\ & & X_t \end{array}, \tag{4.1}$$

in the obvious notation. In what follows we leave this further generalization implicit. To emphasize this structure, we adopt the physics notation $(t, x_s) = x^\mu$ for local "spacetime" coordinates of X.

By way of example, we now fix a base field $\mathbb{F}_q$ and assume we have some basis of physical scalar fields $\phi(t, x_s) = \phi(x)$ taking values in a vector bundle $\mathcal{E}$. Assume also that for each $x \in X$, we have a pairing $v_{ij} : \mathcal{V}_x \times \mathcal{V}_x \to \mathbb{F}_q$ which we can interpret as a "pairing matrix". We can then construct a mapping of the form:

$$\mathcal{V} \times \mathcal{V} \to \mathbb{F}_q \to \mathbb{F}_p \tag{4.2}$$

$$(\phi, \beta) \mapsto v_{ij}\phi^i\beta^j \mapsto \text{Tr}(v_{ij}\phi^i\beta^j), \tag{4.3}$$

where in the last line we used the standard trace map for finite fields.

We can now introduce states $|\phi(x)\rangle$ implicitly defined by the condition:

$$\exp\left(\frac{2\pi i}{p} v_{ij}\widehat{\phi^i}(x)\beta^j(x)\right)|\phi(x)\rangle = \exp\left(\frac{2\pi i}{p} v_{ij}\phi^i(x)\beta^j(x)\right)|\phi(x)\rangle, \tag{4.4}$$

where here, $\beta^j(x)$ is a fixed choice of vector for each stalk $\mathcal{V}_x$, and $\widehat{\phi^i}(x)$ is to be viewed as a quantum operator. We observe that on the righthand side, we have used the standard character map, so $|\phi(x)\rangle$ is an element of a Hilbert space over $\mathbb{C}$. Note that since we are just dealing with characters valued in $\mathbb{C}^*$ we can also define a standard inner product using the usual rules of the path integral formalism.

For example, we can calculate the overlap of field configurations in the "past and future" provided we make the additional assumption that there is a distinguished point $t_i \in X_t$ associated with an initial time, and $t_f \in X_t$ associated with a final time. We can then prescribe fixed boundary conditions for our field at these two times, writing $\phi_i(x_s)$ and $\phi_f(x_s)$. With this in place, we can simply define an overlap of states in the Hilbert space as the quantity obtained by evaluating the path integral in characteristic p:

$$\langle \Phi_f(x_s) | \Phi_i(x_s) \rangle = \int_{\Phi_i}^{\Phi_f} [d\phi] \ \exp(iS[\phi]/\hbar), \tag{4.5}$$

where here, the integral symbol is really an instruction to sum over all the morphisms $X \to Y$, as we have already implicitly discussed. The limits of the integration are to be interpreted as the condition that we prescribe specific values for these morphisms at the fixed values in X_t already indicated.

Correlation functions of operators can now be treated much as one would in the standard characteristic zero case, but we do encounter the subtlety that a notion of "time ordering" does not make much sense, since we have no ordering available for a finite field. In its place, we can speak of repeatedly applying a given operator such as $\exp(2\pi i H/p)$ with H the Hamiltonian, and this composition rule builds up a local notion of past and future (which we expect to break down globally over a finite field). This composition of maps defines a notion of time evolution, much as one would have in iterations of a discretized dynamical system. Of course, at a pragmatic level nothing stops us from just explicitly performing the requisite sum, so we have an operational rule for how to evaluate correlation functions.

One can also see how standard notions of time reversal can work in this setting, at least in basic examples. For example, assuming $X_t = \mathbb{P}^1$ over a finite field $\mathbb{F}_q$, we can consider the map $t \mapsto 1/F(t) = 1/t^q$, where F is the Frobenius conjugation automorphism for $\mathbb{F}_q$. In characteristic zero, we would visualize this as the operation which takes two marked points at 0 and ∞ of $\mathbb{CP}^1$ with two points deleted (i.e. a cylinder.) and interchanges them.

Now, just as in ordinary quantum mechanics, we wish to focus on operators which are unitary. This amounts to the condition that the operators appearing in various exponentiated quantities are really valued in $\mathbb{F}_p$, and not just some characteristic p field. This restriction can be implemented in the above by focusing on $\mathbb{F}_p$-valued exponents, namely by working with the operators:

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}\phi^i(x)\beta^j(x))\right). \tag{4.6}$$

Everything we have said has treated all points in the spacetime on an equal footing. Of course, in a standard physical theory, we expect there to be some redundancy in our basis of states, since for example, we have a notion of time evolution of Cauchy slices. Something similar can be arranged in the characteristic p setting because we can also introduce a

Legendre transform of our Lagrangian, and consequently define a Hamiltonian operator. Doing so, however, requires fixing a notion of a local time coordinate, and so we must make some further assumptions on the geometry of X, as we noted above. Making these choices, we can speak of operators specified at a fixed time, and track their action on the Hilbert space of states now labeled as $|\phi(x_s)\rangle$. None of this is very different from the usual way one discusses quantum fields. This also allows us to discuss, for a fixed time t_0, operators $\Phi^i(x_s) = \phi^i(t_0, x_s)$ valued in V, and a conjugate momentum $\Pi_j(\overrightarrow{x}) = \partial_t \phi^j(t_0, \overrightarrow{x})$ valued in $\mathcal{V}^*$, the dual vector space defined using the canonical pairing v_{ij}. We have already explained how Φ^a acts on states $|\phi(\overrightarrow{x}\rangle$, but now we can also see that Π_j acts as a "translation operation", namely we have:

$$\exp\left(\frac{2\pi i}{p}\Pi_j(\overrightarrow{x})\beta^j(\overrightarrow{x})\right)|\phi(\overrightarrow{x})\rangle = |\phi(\overrightarrow{x}) + \beta(\overrightarrow{x})\rangle. \tag{4.7}$$

Continuing in this way, it should now be clear that even though our physical fields are valued in characteristic p varieties, we are still able to make sense of a physical Hilbert space over the characteristic zero field $\mathbb{C}$ in the standard way.

4.1 Coding Interpretation

Much of our emphasis in this note has centered on transporting physical structures present in characteristic zero over to the case of finite fields. Of course, there is a well-known use for finite fields in the physics literature based on classical and quantum error correcting codes. Here, we turn to an information theoretic interpretation of the previous considerations. Turning the discussion around, one can view the present section as an attempt to build up the previously encountered physical structures from a purely information theoretic starting point. In this section we make use of the structure of classical codes generated from algebraic curves over finite fields, as well as their extension to quantum error correcting codes. To keep the discussion streamlined, we have deferred some standard definitions to Appendix K which contains additional details. See figure 2 for a depiction of encoding via schemes over finite fields.

As a first step in this direction, suppose we consider a scalar field theory, where we take our spacetime X to be the projective line over the finite field $\mathbb{F}_q$, and our physical field is a rational map $\phi : X \to Y$ with Y the affine line. We mark the "point at infinity" in X and specify a prescribed pole structure at this location. Given this, we can interpret X as a single timelike direction, and we can use our setup from section XXX to construct a Hilbert space of states. This requires us to also specify a notion of a canonical pairing v_{ij} which we implicitly use to raise and lower indices. In the Heisenberg picture, we label these states as $|\phi\rangle$, where ϕ is interpreted as taking values in $\mathbb{F}_q$, where we introduce at some fixed time t_* an operator Φ^i and its conjugate momentum $\Pi_j = \partial_t \phi_j|_{t=t_*}$, with the properties that for

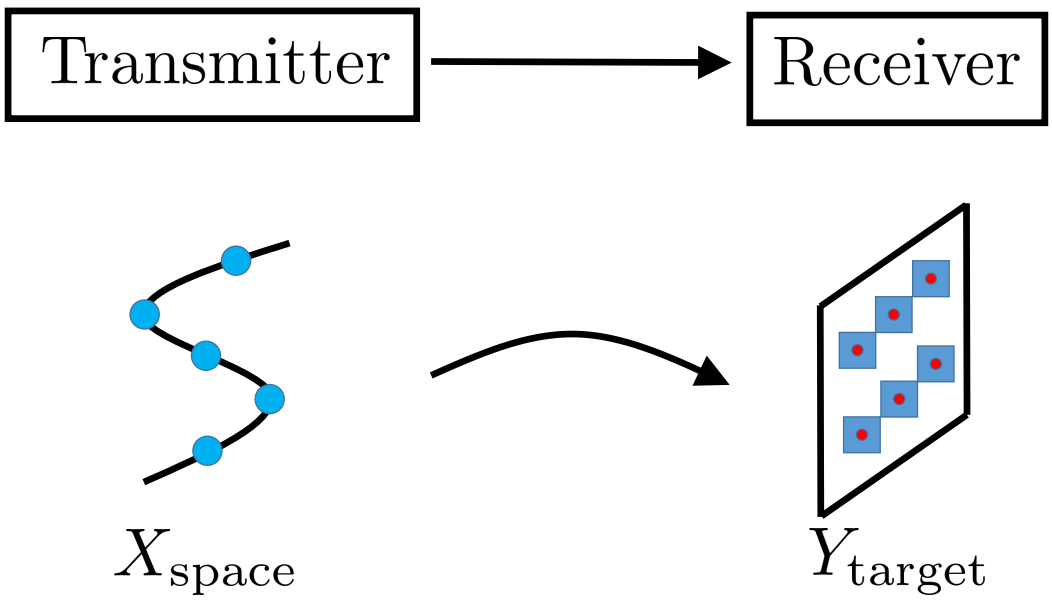

Figure 2: Depiction of how a spatial geometry furnishes a code, both in the classical and quantum setting. The encoding of information is achieved via maps from a transmitter (source) to a receiver (target space).

$a, b \in \mathbb{F}_q$ we have:

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}a^i\Phi^j)\right)|\phi\rangle = \exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}a^i\phi^j)\right)|\phi\rangle \tag{4.8}$$

$$\exp\left(\frac{2\pi i}{p}b^j\Pi_j\right)|\phi\rangle = |\phi + b\rangle\,, \tag{4.9}$$

so we recognize that these are building up standard qudit error operations which we denote as:

$$E_{ab} = T_a R_b \tag{4.10}$$

with:

$$T_a = \exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}a^i\Phi^j)\right), \quad R_b = \exp\left(\frac{2\pi i}{p}b^j\Pi_j\right), \tag{4.11}$$

so in other words, we get a single qudit in the case of a quantum mechanical system defined in this way. Turning the discussion around, we see that the set of qudit error operators provides an implicit definition of a quantum mechanical system in characteristic p.

We can generalize this to something closer to the case of a field theory by now assuming our spacetime X takes the form of $X_{\text{time}} \times X_{\text{space}}$, where again we assume X_{time} is a projective line, and X_{space} is now taken to be a smooth projective curve over the base field $\mathbb{F}_q$. In this case, we again take Y to be an affine line, so we might as well view ϕ as an element of the line bundle $\mathcal{L}(G)$, with G a divisor indicating the "points at infinity" of X. From what we have

said above, we should also view the a's and b's appearing in our discussion above as elements of the line bundle $\mathcal{L}(G)$. We can now build a set of linear codes by considering n distinct points $P_1, ..., P_n$ of X_{space} as well as the divisor $D = P_1 + ... + P_n$. Evaluating at these points, we get the classical linear code $C_{\mathcal{L}}(D, G)$. Of course, in our path integral prescription we perform this evaluation, but in a slightly more involved away, first constructing an evaluation map to the action, and then mapping this to a character of our finite field. The point remains, however, that at least in this simplified setting, each of our physical field configurations can be viewed as an element of the Riemann-Roch space.

Having seen how classical linear codes naturally emerge from this setting, we can ask about the construction of quantum stabilizer codes. As reviewed in Appendix K below line (K.35), one of the standard results is to actually specify an $\mathbb{F}_p$ linear subspace $C \subset \mathbb{F}_q^{2n}$ which is self-orthogonal with respect to the standard symplectic pairing. In the present context, we consider the simplifying situation where C is actually an $\mathbb{F}_q$-linear subspace of dimension k. The precise idea is to specify the duals to the vectors a and b as elements $(\Phi(P_l); \Pi(P_l))$ evaluated at the points in X_s specified by the divisor $D = P_1 + ... + P_n$. Here, the index $l = 1, ..., n$ runs over the evaluation points, and we have left implicit the vector index. The main thing we need to establish is that our phase space builds a self-orthogonal space with respect to the standard symplectic pairing. For ease of exposition, we assume that we can change basis so that the v_{ij} appearing earlier is just the identity matrix and we assume the standard relation:

$$\Pi = \partial_t \Phi, \tag{4.12}$$

in the obvious notation. Now, given two elements $(\Phi; \Pi)$ and $(\Phi'; \Pi')$ of C, we observe that the symplectic pairing is:

$$(\Phi; \Pi) *_s (\Phi'; \Pi') = \text{Tr}(\Phi \cdot \Pi' - \Pi' \cdot \Phi). \tag{4.13}$$

But, via equation (4.12), we observe that $\Pi' = \partial_t \Phi'$ and $\Pi = \partial_t \Phi$. By inspection of equation (4.13), the condition that the code is self-orthogonal with respect to $*_s$ now follows. Treating these classical evaluation points as the span of possible values of the a and b appearing in the CSS construction [49, 50], the theorem of reference [51] stated below line (K.35) now gives us an $[[n, n-k, d(C^{(s)} \backslash C)]]_q$ quantum stabilizer code.

We note that from the perspective of symplectic geometry in characteristic zero, all that we have done is exploit the appearance of a Lagrangian submanifold in the symplectic phase space, i.e. a middle dimensional subspace which provides a canonical split between positions Φ and conjugate momenta $\Pi = \partial_t \Phi$. The characteristic p analog of this statements provides us with our construction of a self-orthogonal linear subspace.

The above statements also generalize to the case where we work with more general sorts of classical codes. As we already mentioned in Appendix K near line (K.18), we can also consider situations where we have a stable vector bundle E over a curve X. We will return to examples of such structures in section 7. Recall from our discussion in Appendix K, a

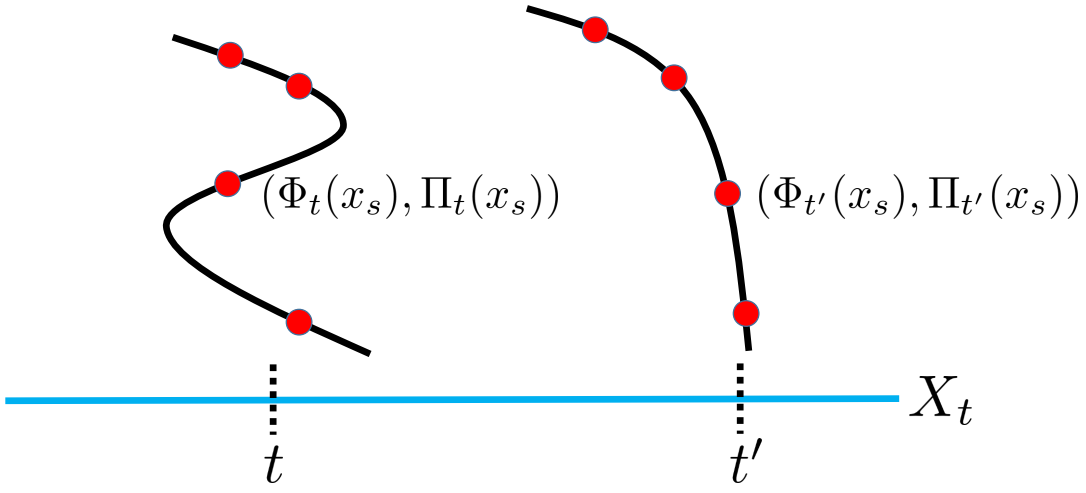

Figure 3: Depiction of the motion of the trajectory of a field $\Phi_t(x_s)$ and its conjugate momentum $\Pi_t(x_s)$ in phase space as a function of the time coordinate in X_{time}. This phase space structure provides a self-orthogonal space with respect to the symplectic pairing $*_s$, and thus can generate a quantum stabilizer code.

rank r vector bundle allows us to specify a code subspace in $\mathbb{F}_Q^n$, where $Q = q^r$. While this is not always an $\mathbb{F}_Q$ linear space, it is $\mathbb{F}_q$ linear. Consequently, we can use the same sort of "phase space argument" used above in the case of line bundles (where $r = 1$), we then get an $[[n, n-k, d]]_Q$ quantum stabilizer code.

From the above considerations, we thus see a different physical interpretation of our construction of field theories in characteristic p. In particular, we can also view our path integral as performing a sum over possible classical codewords (after composing with evaluation maps) and the resulting eigenspaces generated by the resulting quantum error correcting codes define a basis of states in a physical Hilbert space. Observe that in constructing this Hilbert space, even if we have roughly n physical points (as dictated by the order of the divisor D in X_s), the actual information content is instead "delocalized" across several points.

4.2 Quasi-Locality

There is one more layer of abstraction which naturally fits with our discussion, and suggests how the structure of locality emerges from a somewhat more primitive construct. At a rough level, we view this as stating that our formalism for defining path integrals in characteristic p amounts to specifying a topos of quantum stabilizer codes. Indeed, there is some suggestive overlap with potential uses of topoi in physics discussed in references [10–15] which might be interesting to develop further. That being said, we will not attempt a match with the

considerations found there.

To set the stage, we recall that one of the awkward features of algebraic geometry in general is the rather course nature of the Zariski topology. In characteristic zero, one can often supplement this by treating various spaces as real or complex analytic, but in working with finite fields there is always an intrinsic discretization to the resulting system, a point which we are actually attempting to exploit. Nevertheless, there is a sufficiently rich notion of topology we can introduce which allows one to construct non-trivial cohomology theories. We review some features of the resulting topologies in Appendix J. The main point is that to get a suitable notion of "coverings by open sets" it is important to emphasize more the collections of morphisms to a given scheme.

In our context, we have already seen that there is a sense in which the spacetime X as well as the target space Y can be equipped with suitable topoi. From the perspective of coding theory, we have also interpreted this as a general transmission problem in information theory. From an ambitious standpoint, one might view the associated topologies as defining a construct even more primitive than a Hilbert space, the latter only appearing after further processing in the language of quantum error correcting codes.

Of course, if we ever wish to return to the world of observation, we must somehow find a path back from characteristic p geometry to the more familiar terrain of physics in characteristic zero. We turn to these issues later in section 8.

5 Fermionic Systems

Our discussion up to this point has focussed on systems involving bosonic degrees of freedom. In this section we develop a parallel story for fermionic degrees of freedom. We shall make use of the main geometric elements for bosonic systems developed previously. For now, we again restrict to the special case where:

$$\hbar = \frac{p}{2\pi}, \tag{5.1}$$

with p a prime. For some earlier discussions of arithmetic with Grassmann algebras see e.g. [52, 53].

As before, we motivate our analysis by beginning with a quantum mechanical system with discretized observables. In this case, we consider a two state Hilbert space spanned by the states $|\uparrow\rangle$ and $|\downarrow\rangle$, and introduce real fermionic operators $\widehat{\psi}$ and $\widehat{\chi}$ which satisfy the algebra:

$$\left\{\widehat{\psi}, \widehat{\chi}\right\} = 1, \quad \widehat{\psi}^2 = \widehat{\chi}^2 = 0. \tag{5.2}$$

These operators act on our states as follows:

$$\widehat{\psi}\,|\downarrow\rangle = 0, \quad \widehat{\psi}\,|\uparrow\rangle = |\downarrow\rangle \tag{5.3}$$

$$\widehat{\chi}\,|\downarrow\rangle = |\uparrow\rangle\,, \quad \widehat{\chi}\,|\uparrow\rangle = 0. \tag{5.4}$$

As an example, we can consider the Hamiltonian operator:

$$\widehat{H} = m\widehat{\chi}\widehat{\psi}. \tag{5.5}$$

Time evolution of states is accomplished by acting with the unitary operator:

$$U(t) = \exp(-i\widehat{H}t/\hbar), \tag{5.6}$$

where we assume (as discussed previously) that we can only make measurements in a smallest time step $t \in \mathbb{Z}$. Even though the fermionic degrees of freedom are already discretized, one might ask whether there are any restrictions on the parameter m. Observe that the explicit form of our time evolution operator is:

$$U(t) = \exp\left(-\frac{2\pi i}{p} m\widehat{\chi}\widehat{\psi}t\right). \tag{5.7}$$

Acting on the two states, we have:

$$U(t)\,|\downarrow\rangle = \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{2\pi i}{p} m\widehat{\chi}\widehat{\psi}t\right)^n |\downarrow\rangle = 0 \tag{5.8}$$

$$U(t)\left|\uparrow\right\rangle = \sum_{n\geq 0}\frac{1}{n!}\left(-\frac{2\pi i}{p}m\widehat{\chi}\widehat{\psi}t\right)^n\left|\uparrow\right\rangle = \exp(-\frac{2\pi i}{p}mt)\left|\uparrow\right\rangle. \tag{5.9}$$

Provided we restrict m to the integers, we see that the complex phase for $|\uparrow\rangle$ will eventually return after at most p time steps. With this motivation in mind, we can now proceed to develop the parallel formalism for fermionic path integrals. This is an entirely standard development in characteristic zero, and is covered in detail for example in [54] and Appendix A of [55].

What is not so standard is to understand the characteristic p version of fermionic systems.Here, we will aim to convey the main physical issues. The first issue we face is that we will need to supplement the finite field $\mathbb{F}_p$ by anti-commuting Grassmann variables. We define Grassmann variables χ_i by requiring that they anti-commute, i.e.:

$$\chi_i\chi_j = -\chi_j\chi_i. \tag{5.10}$$

The appropriate notion of an $\mathbb{F}_p$-valued Grassmann variable in this setting will be that the extension of the Frobenius endomorphism F to anticommuting variables leaves such variables fixed. With this in mind, we require that an $\mathbb{F}_p$-Grassmann variable χ satisfies:

$$F(\chi) = \chi, \tag{5.11}$$

which is the analog of Hermitian conjugation in characteristic zero. We also demand that for any bosonic $\phi \in \mathbb{F}_p$ that we have:

$$F(\phi\chi) = \phi\chi. \tag{5.12}$$

Now, given multiple $\mathbb{F}_p$ Grassmann variables, we would like to extend the action of the Frobenius automorphism to products of Grassmann variables.

A priori, there are two ways in which one might attempt to proceed. On the one hand, if we insist on keeping all coefficients valued in $\mathbb{F}_p$, we can consider an action which respects multiplicative order. On the other hand, we can allow the Frobenius map to switch the order of fermions:

$$F_{\text{option 1}}(\chi\psi) = F(\chi)F(\psi) = \chi\psi, \tag{5.13}$$

$$F_{\text{option 2}}(\chi\psi) = F(\psi)F(\chi) = \psi\chi. \tag{5.14}$$

In the physical setting, it is more natural to treat Grassmann fields as operators, so a conjugation operation would switch the order of multiplication. We therefore focus on "option 2." The case of "option 1" is of interest in its own right, however, and we discuss how to build supersymmetric actions with this choice in Appendix C. One can view the two procedures as related by analytic continuation, i.e. by multiplying some fields by appropriate "imaginary numbers."

Now, in characteristic zero, we are often interested in Hermitian operators built from products of such fermionic fields. The way we do this involves multiplication by factors of $i = \sqrt{-1}$, since complex conjugation reverse the order of multiplication on Grassman fields and $i^* = -i$. We need to introduce a suitable notion of "i" which flips sign under Frobenius conjugation. We already encountered this feature in our discussion of vector potentials in section 3.4, where we noted that $\sqrt{-1}$ sometimes will not accomplish this goal. For example, in $\mathbb{F}_5$, observe that $3^2 = -1$. Just as in section 3.4, we will instead seek out a root of the polynomial equation:

$$x^p = -x, \tag{5.15}$$

and we denote one such root by $\widehat{i}$. Observe that by design, we have:

$$F(\widehat{i}) = \widehat{i}^p = -\widehat{i}. \tag{5.16}$$

Since $\widehat{i}$ is not invariant under Frobenius conjugation, it is not an element of $\mathbb{F}_p$. Note, however, that its square $\widehat{i}^2$ is invariant, and is therefore an element of $\mathbb{F}_p$. What we cannot assert, however, is that $\widehat{i}^2 = -1$. Indeed, $\widehat{i}$ is an element of $\mathbb{F}_q$ with $q = p^2$. A combination of $\mathbb{F}_p$-valued Grassmann numbers invariant under Frobenius conjugation can now be obtained through a product such as:

$$\widehat{i}\chi\psi. \tag{5.17}$$

Having set our conventions for Grassmann coordinates in characteristic p, we can now proceed to build fermionic actions. As a warmup, we first develop the 1D path integral. Introduce a formal parameter t and expand our fermionic fields via the power series:

$$\chi(t) = \sum_i \chi_i t^i \quad \text{and} \quad \psi(t) = \sum_i \psi_i t^i, \tag{5.18}$$

where each of the coefficients is an $\mathbb{F}_p$-Grassmann variable. Returning to our two state system, the Lagrangian will be viewed as a Grassmann even polynomial in the variable t, and the action is obtained through the evaluation map:

$$S = \sum_{t \in X} \left(\widehat{i}\chi\partial_t\psi - \widehat{i}m\chi\psi \right). \tag{5.19}$$

Observe that each term in the above sum is invariant under the Frobenius map, and should thus be viewed as $\mathbb{F}_p$ valued.

Evaluation of the path integral now proceeds just as in characteristic zero; We can perform Grassmann integrals by expanding the exponentials, and evaluate fermionic correlation functions in the standard way.

This generalizes to other spacetime dimensions. With conventions as in subsection 3, we introduce a polynomial ring $\mathbb{F}_q[t_1, ..., t_D]$ for our bosonic physical fields. We can supplement this by tensoring with a set of Grassmann coordinates. Along these lines, recall from equation

(3.4) that a bosonic physical field was initially presented as a power series expansion:

$$\phi(t_1, ..., t_D) = \sum_{i_1,...,i_D} \phi_{i_1...i_D}(t_1)^{i_1}...(t_D)^{i_D}, \tag{5.20}$$

We can write a fermionic analog of this by expanding with Grassmann valued coefficients:

$$\chi(t_1, ..., t_D) = \sum_{i_1,...,i_D} \chi_{i_1...i_D}(t_1)^{i_1}...(t_D)^{i_D}, \tag{5.21}$$

where each coefficient $\chi_{i_1...i_D}$ is to be treated as a Grassmann coordinate. Now, in the bosonic case, the path integral instruction is to sum over all these choices of $\phi_{i_1...i_D}$. In the fermionic context, we perform a Grassmann integral. So, we can again construct Lagrangians and actions for our physical fields. The only difference now is that there will be some Grassmann dependence. The main condition we impose is that the coefficients of any expression in our action are again $\mathbb{F}_p$ valued. Again, this is the analog of a "reality condition" in the characteristic p context.

What sort of correlation functions should we consider computing in this context? As in the case of purely bosonic systems, we observed that operators which respect our reduction modulo p are the ones of interest. In the fermionic context, the standard expansion of Grassmann integrals might suggest that this is not possible. Of course, in quantum field theory we are accustomed to viewing operators constructed from composite fermions as bosonic objects. This in turn means that in this setting, the simplest class of operator correlation functions to consider are those which are built from such bosonic operators. An example of this sort is the time evolution operator of our two level system introduced in equation (5.6).

Proceeding along the same steps following for our bosonic field theory, we can extend all of these considerations to far more general spacetimes X and target spaces Y. In this more general setting, it is appropriate to replace our polynomials by expressions which are locally rational functions. This is acceptable provided we specify what happens at the singularities of the evaluation map.

Now, up to this point we have ignored the spin of our fermionic degrees of freedom. In characteristic zero, one can locally speak of a spinor bundle, and in suitable circumstances this extends to the global manifold. In more algebraic terms, we can introduce a sheaf of spinors $\mathcal{S}$ such that along each stalk $\mathcal{S}_x$, we have a spinor representation of the Lorentz algebra.

To carry out the same sort of construction in characteristic p, we first need to decide on a suitable notion of an orthogonal group. Fixing a symmetric bilinear form, η_{ab}, we can again speak of linear transformations which leave this bilinear form invariant. We refer to the corresponding Lie algebra as $\mathfrak{spin}(\eta)$. We can then construct finite-dimensional irreducible representations of the corresponding Lie algebra, which we interpret as specifying the "spin"

of the corresponding physical field. We define a spinor sheaf as one in which for each stalk $\mathcal{S}_x$ there is a natural group action by $\mathfrak{spin}(\eta)$. It is in this sense that we are able to define spinors. From this perspective, the evaluation of each fermionic field at a point $x \in X$ should be viewed as being valued in $\mathcal{S}_x$. Again, this is quite analogous to what happens in characteristic zero.

5.1 Supersymmetry

From the way we have set up our action principle, we can even entertain a notion of supersymmetry which interchanges bosonic and fermionic degrees of freedom. Note that in lattice supersymmetry [56], there are some difficulties because finite difference operations do not respect a Leibniz rule, and this is crucial in satisfying the standard supersymmetry algebra [57]. Here, we are working in terms of general rational polynomials, and so the usual "rules of the game" for supersymmetry should carry through, at least in constructing supersymmetric actions.

To illustrate, we construct a characteristic p supersymmetric quantum mechanics. It is superficially rather close in form to the one in characteristic zero, but there are some important subtleties having to do with factors of "i".

As a warmup, we briefly review the case of $\mathcal{N} = 2$ supersymmetric quantum mechanics in characteristic zero. In that setting, the Lagrangian is:

$$L = \frac{1}{2}(\partial_t \phi)^2 + i\overline{\Psi}\partial_t \Psi + \frac{1}{2}f^2 + W'f + W''\overline{\Psi}\Psi. \tag{5.22}$$

Here, ϕ is a real bosonic field, $\Psi = \psi_1 + i\psi_2$ and $\overline{\Psi} = \psi_1 - i\psi_2$ are complex Grassman fields, f is a real auxiliary field, and $W(\phi)$ is a superpotential. Additionally, we have adopted the standard physics convention which is to reverse the order of Grassmann variables under complex conjugation. From this, we see that the Lagrangian is invariant under complex conjugation. The action is invariant under two supersymmetry transformations (see e.g. [58] for a recent discussion):

$$\delta_1 \phi = i\Psi, \quad \delta_1 \Psi = 0, \quad \delta_1 \overline{\Psi} = -(\partial_t \phi + if), \quad \delta_1 f = -\partial_t \Psi \tag{5.23}$$

$$\delta_2 \phi = i\overline{\Psi}, \quad \delta_2 \Psi = -(\partial_t \phi - if), \quad \delta_2 \overline{\Psi} = 0, \quad \delta_2 f = +\partial_t \overline{\Psi}. \tag{5.24}$$

Let us verify that the action is invariant under these two transformations. Under δ_1, we have:

$$\begin{aligned}
\delta_1 L &= (\partial_t \phi)\,(i\partial_t \Psi) + i(-\partial_t \phi - if)\partial_t \Psi + (-\partial_t \Psi)f && (5.25)\\
&\quad + W''(i\Psi)f + W'(-\partial_t \Psi) && (5.26)\\
&\quad + W''(-\partial_t \phi - if)\Psi && (5.27)\\
&= \partial_t\,(-W'\Psi)\,, && (5.28)
\end{aligned}$$

which is a total derivative. Assuming suitable boundary conditions for our path integral, we verify that supersymmetry is a symmetry of the system. Consider next the variation under δ_2. This yields:

$$\begin{aligned}\delta_2 L &= (\partial_t\phi)(i\partial_t\overline{\Psi}) - i\partial_t(-\partial_t\phi + if)\overline{\Psi} + f(\partial_t\overline{\Psi}) && (5.29)\\ &+ W''(i\overline{\Psi})f + W'\partial_t\overline{\Psi} && (5.30)\\ &- W''(-\partial_t\phi + if)\overline{\Psi} && (5.31)\\ &= \partial_t((\partial_t\phi)(i\partial_t\overline{\Psi}) + f(\partial_t\overline{\Psi}) + W'\overline{\Psi}). && (5.32)\end{aligned}$$

Note the appearance of the minus signs. This is because our convention is to only vary the "leftmost" fermionic field. We stress that nothing depends on this choice. As is standard, we can integrate out the auxiliary field f, and arrive at a physical potential for the field ϕ given by:

$$V(\phi) = \frac{1}{2}W'W'. \tag{5.33}$$

We now turn to the characteristic p version. As we already mentioned, we assume that Frobenius conjugation reverses the order of multiplication for Grassmann fields. This means that up to "some factors of i," the structure of our action should look rather similar.

With this in mind, we now consider a single $\mathbb{F}_p$ valued bosonic field $y(t)$ and a pair of $\mathbb{F}_p$ valued Grassmann variables $\chi(t)$ and $\psi(t)$. We could in principle introduce a "complex field" $\Psi = \chi + \widehat{i}\psi$ as well, but to track the $\mathbb{F}_p$ structure explicitly, we have chosen the current presentation. We also introduce an $\mathbb{F}_p$ valued auxiliary field $f(t)$ and a superpotential $W(\phi)$ which will be a polynomial in the ϕ variable with coefficients in $\mathbb{F}_p$. We denote the derivatives of W with respect to ϕ as W' and W''. Our proposed Lagrangian is:

$$L = \frac{1}{2}(\partial_t\phi)^2 + \widehat{i}\chi\partial_t\psi - \frac{\widehat{i^2}}{2}f^2 + W'f + \widehat{i}W''\chi\psi. \tag{5.34}$$

We now verify that this Lagrangian is supersymmetric. We introduce the two variations:

$$\delta_1\phi = \widehat{i}\psi, \quad \delta_1\psi = 0, \quad \delta_1\chi = -(\partial_t\phi + \widehat{i}f), \quad \delta_1 f = -\partial_t\psi \tag{5.35}$$

$$\delta_2\phi = \widehat{i}\chi, \quad \delta_2\psi = -(\partial_t\phi - \widehat{i}f), \quad \delta_2\chi = 0, \quad \delta_2 f = +\partial_t\chi. \tag{5.36}$$

Consider first varying with respect to δ_1. This yields:

$$\begin{aligned}\delta_1 L &= (\partial_t\phi)\left(\widehat{i}\partial_t\psi\right) + \xi(-\partial_t\phi - \widehat{i}f)\partial_t\psi - \widehat{i^2}(-\partial_t\psi)f && (5.37)\\ &+ W''(\widehat{i}\psi)f + W'(-\partial_t\psi) && (5.38)\\ &+ W''(-\partial_t y - \widehat{i}f)\psi && (5.39)\\ &= \partial_t\left(-W'\psi\right). && (5.40)\end{aligned}$$

Observe that we have a "total derivative". As far as we are aware, there is no characteristic p analog of Stokes' theorem, but we shall interpret the presence of such terms as physically innocuous. Our reason for doing so is that in any sensible physical formulation, we would need to define an action modulo exact differential forms anyway, and differential forms do make sense in characteristic p.

Next, consider varying with respect to δ_2. This yields:

$$\delta_2 L = (\partial_t \phi)\left(\widehat{i}\partial_t \chi\right) - \widehat{i}\partial_t(-\partial_t\phi + \xi f)\chi - \widehat{i^2}(+\partial_t\chi)f \tag{5.41}$$

$$+ W''(\widehat{i}\chi)f + W'(+\partial_t\chi) \tag{5.42}$$

$$- W''(-\partial_t\phi + \widehat{i}f)\chi \tag{5.43}$$

$$= \partial_t((\partial_t\phi)\,\widehat{i}\chi - \widehat{i^2}f\chi + W'\chi), \tag{5.44}$$

which is again a "total derivative." Integrating out the auxiliary field f, we arrive at a potential for the field ϕ given by:

$$V(\phi) = \frac{1}{2}W'W', \tag{5.45}$$

We can extend this analysis in a number of ways. For one, we can consider multiple fields ϕ^A, χ^A, ψ^A and f^A. Following our discussion of section 3, we can interpret this as $\mathbb{F}_q$ valued fields. In this case, the condition that we produce an $\mathbb{F}_p$ valued action is satisfied by choosing an $\mathbb{F}_q$ valued function $w(\phi)$ and then taking its norm to build the superpotential:

$$W = \prod_{i=0}^{n} F^i(w(\phi)) = w(\phi)^{1+p+\ldots+p^{n-1}} = w(\phi)^{(1-p^n)/(1-p)} \tag{5.46}$$

Observe that a critical point of W is necessarily either a zero or a critical point of $w(\phi)$. We can also introduce more general kinetic terms, much as we would in the characteristic zero setting. For example, we can write:

$$L = \frac{1}{2}K_{AB}\partial_t\phi^A\partial_t\phi^B + \widehat{i}K_{AB}\chi^A\partial_t\psi^B - \frac{\widehat{i^2}}{2}K_{AB}f^Af^B + \frac{\partial W}{\partial\phi^A}f^A + \widehat{i}\frac{\partial^2 W}{\partial\phi^A\partial\phi^B}\chi^A\psi^B. \tag{5.47}$$

5.2 A Cohomology Theory

This discussion also allows us to set up a physically motivated cohomology theory. Working on shell so that:

$$f^A = \widehat{i^{-2}}K^{AB}\frac{\partial W}{\partial y^B} \tag{5.48}$$

the supercharges are given by:

$$Q_+ = \widehat{i}\psi^A \left(\frac{\partial}{\partial \phi^A} + \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) = - \left(\partial_t \phi^A + \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) \frac{\partial}{\partial \chi^A} \tag{5.49}$$

$$Q_- = \widehat{i}\chi^A \left(\frac{\partial}{\partial \phi^A} - \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) = - \left(\partial_t \phi^A - \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) \frac{\partial}{\partial \psi^A}, \tag{5.50}$$

where we have indicated by an explicit "derivative" (as dictated by the conjugate momentum) how it acts on a given field. We observe that both Q's are nilpotent:

$$Q_+^2 = Q_-^2 = 0 \tag{5.51}$$

and so can be used to define cohomology theories in characteristic p. In this setting, the Q's act on the space of superfield configurations. The natural grading is specified by the Fermion number, namely the number of Grassmann fields.

Now, in characteristic zero, there is a close interplay between Q-cohomology and other well known cohomological theories such as de Rham and Dolbeault cohomology. Here, the situation is quite a bit more subtle because in characteristic p, we do not have the analog of the Poincaré lemma which ensures that in suitably "small" patches that at least locally, any differential form can be written as an exact differential form.

A reasonable analog in characteristic p to the characteristic zero de Rham cohomology goes under the name of crystalline cohomology (see e.g. [59–61]).[7] The main idea is to find a suitable way to "thicken" a characteristic p variety so as to get an analog of the Poincaré lemma. This proceeds by generating a lift of a given scheme to a characteristic zero variety. Given this, it is tempting to posit that the Q-cohomology we have just specified will work in a similar fashion.

Indeed, we note that our actual starting point for constructing physical fields began by dealing with integer valued fields, so we are free to return to this setting. Given a field ϕ taking values in $\mathbb{Z}$, we can consider its presentation in terms of a p-adic integer in $\mathbb{Z}_p$ via the formal expansion:

$$\phi = \sum_{i \geq 0} \phi_i p^i \tag{5.52}$$

in terms of the Teichmüller representatives ϕ_i (see Appendix G). To compute actual cohomologies, we can get a "first approximation" by working modulo p. Then, we can refine this approximation by working modulo p^2, and so on. The more formal way to state this is that we view our integer valued field as specifying a Witt vector, and then addition and multiplication of physical fields is treated as the corresponding operation on $\mathbb{W}$, the space of Witt vectors (see Appendix G). There is a natural reduction mod p^n so we can also speak

[7] In the context of the Weil conjectures, it is actually more common to consider étale and ℓ-adic cohomology theories, but on physical grounds we expect crystalline cohomology to also provide an appropriate framework as well.

of $\mathbb{W}_n = \mathbb{W}/p^n\mathbb{W}$. Giving a full account of crystalline cohomology would take us to far afield. The main point for us is that in many cases of interest, we can consider a related characteristic zero scheme Z over $\mathbb{W}$. In this setting, we can indeed work in terms of de Rham cohomology, and thus obtain the relation between the crystalline cohomology of a X over a field k and its characteristic zero "cousin" Z:

$$H^i_{\text{cris}}(X/\mathbb{W}) = H^i_{\text{DR}}(Z/\mathbb{W}). \tag{5.53}$$

These cohomologies are in turn constructed via the inverse limits:

$$H^i_{\text{cris}}(X/\mathbb{W}) = \varprojlim H^i_{\text{cris}}(X/\mathbb{W}_n) \tag{5.54}$$

$$H^i_{\text{DR}}(Z/\mathbb{W}) = \varprojlim H^i_{\text{DR}}(X/\mathbb{W}_n). \tag{5.55}$$

In fact, it has also been appreciated that there are some limitations to using crystalline cohomology. One issue is that the theory makes the most sense when X is smooth and proper over a base field k. To handle the more general situation, one often deals with a generalization known as rigid cohomology which can be applied in a more general setting [62] (see the lecture slides of reference [63] as well as the book [64]). The important point for us is that this defines a universal p-adic Weil cohomology theory, and admits comparison theorems to de Rham cohomology (just like the crystalline case). Since our supersymmetric quantum mechanics formulation does not really require a smooth variety, it is tempting to conjecture that the Q-cohomology we have been dealing with specifies a crystalline cohomology in the smooth case:

$$H^i_Q(X) \simeq H^i_{\text{cris}}(X/\mathbb{W}), \tag{5.56}$$

while in the more general setting, we expect:

$$H^i_Q(X) \simeq H^i_{\text{rig}}(X). \tag{5.57}$$

Part of establishing such a correspondence will of course entail being more precise about the ring of coefficients for these different situations.

Now, in the physical theory, we often view the Q-cohomology as elements in a finite-dimensional Hilbert space. From the above considerations, it would seem natural to restrict the coefficients of this Hilbert space to a field of characteristic zero such as the one used in defining crystalline cohomology. The appearance of a Hilbert space also allows us to define an index for Q, as given by (see e.g. [18]):

$$\text{Ind}Q = \text{Tr}(-1)^{\mathbf{F}} = \text{ker}Q - \text{coker}Q \tag{5.58}$$

where $\mathbf{F}$ is the fermion number operator. At this point, an important comment is that even though we are dealing with a discretized spacetime and target space, we are considering all possible morphisms between these spaces, as well as their lift to formal characteristic zero

spaces. For this reason we should expect on general grounds that the index of Q is in general non-trivial. This again distinguishes the present approach from lattice formulations.

By design, none of this is very different from supersymmetry in characteristic zero. Now, there is a rich mathematical story for supersymmetric quantum mechanics [65,66]. It would be interesting to see whether this carries over to the present setting. Here we set our ambitions lower and apply this in the most simple-minded way, observing that for Wess-Zumino models, the index of Q just counts the critical points of the superpotential W, namely the locus where $dW = 0$. Geometrically, we started with a target space Y, and now can specify this subvariety as $\{dW = 0\} \subset Y$. As a a simple application, if we write $w = \widetilde{y}h(y)$ with $\{h = 0\} = Z$ a smooth subvariety of Y, then $\widetilde{y}$ serves as a Lagrange multiplier so that the critical points of w correspond to all the points of Z. Again, we emphasize that these notions continue to make sense in characteristic p and make no reference to any metric (which is important since we do not have a metric!). Our cohomology theories provide a convenient way to compute the resulting point set as a set of "vacua" associated with vanishing potential energy.

5.3 Zeta Functions

We now turn to a few brief comments on the connection between Zeta functions in characteristic p and our supersymmetric quantum mechanics. We introduced an index which counts (with signs) "vacua," or more precisely the critical points of a superpotential $W(y)$. The zero locus defines a variety V in characteristic p, and we can consider varying the base field $\mathbb{F}_q$, which as we have seen corresponds to adding more particles into the system. From our earlier remarks on interpreting physics on $\mathbb{F}_q/\mathbb{F}_p$ for $q = p^n$ as defined by a system of n particles, we also see that there is a natural action of the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \simeq \mathbb{Z}/n\mathbb{Z}$ on this system of particles. We note that this is generated by the q^{th} Frobenius map $x \mapsto x^q$. To account for this redundancy, it is appropriate to actually only count contributions to the index up to this group action. There is of course the subtlety that this group action may not act transitively on the space of solutions, but this is simply the price we pay in setting up the appropriate particle statistics. Introducing a fugacity z to track the number of particles, we introduce the more general formula:

$$\sum_{n\geq 1} \mathrm{Tr}_n \left((-1)^{\mathbf{F}} z^n\right) = \sum_{n\geq 1} \frac{\#V(\mathbb{F}_{p^n})}{|\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)|} z^n = \sum_{n\geq 1} \#V(\mathbb{F}_{p^n}) \frac{z^n}{n} = \log Z_{V,p}(z), \tag{5.59}$$

which we recognize as the log of the celebrated Hasse-Weil Zeta function in characteristic p. An additional remark is that we can of course change the base field from $\mathbb{F}_p$ to $\mathbb{F}_q$, and this also has a clear interpretation in our setting.

In Appendix I we collect a few examples of Zeta functions. In some cases, we can evaluate these expressions "by hand," but the more general case requires quite a bit more machinery. As some simple examples, we can see that in the special case where V is the affine line, we

get, via our superpotential computation:

$$\#\mathbb{A}^1(\mathbb{F}_{p^n}) = p^n, \tag{5.60}$$

while in the case of the projective line, we get:

$$\#\mathbb{P}^1(\mathbb{F}_{p^n}) = 1 + p^n. \tag{5.61}$$

We observe that the Zeta function in these two cases are related to the partition functions of free particles. For example, we have:

$$Z_{\mathbb{A}^1,p}(z) = \frac{1}{1-pz} \tag{5.62}$$

$$Z_{\mathbb{P}_1,p}(z) = \frac{1}{(1-z)(1-pz)}. \tag{5.63}$$

This basically parallels how one would expect to apply the standard Weil cohomology theories to compute the Zeta function. For example, in both the case of étale, ℓ-adic, (see [17]) and rigid (see [67]) cohomology theories, one first calculates the cohomology groups $H^i(V)$ (we ignore subtleties with the coefficient ring) and then specifies the induced action of the Frobenius map $\psi : V \to V$, associated with the pullback $\psi^* : H^i(V) \to H^i(V)$. One can then count the fixed points of the Frobenius map via the associated signed index formula, namely via a formula such as:

$$\#\mathrm{Fix}(\psi) = \sum_i (-1)^i \mathrm{Tr}(\psi^*, H^i(V)). \tag{5.64}$$

Indeed, we are performing the same set of operations in our physical setting, up to one subtlety. Observe that our Q-cohomology can be viewed as specified with respect to a coefficient ring in the p-adic integers. That being said, since we are talking about computing a supersymmetric index with physical states in a standard Hilbert space, we seem to instead be referencing coefficients in $\mathbb{C}$. As we discuss in section 8, the path integral is really furnishing us with characters valued in a $\mathbb{C}$ as obtained from a "henselization" (see Appendix G) of the integers embedded in the p-adics. Because of this, the counting problems really do appear to be the same. All this is to say the usual physical strategy for computing the supersymmetric index appears to line up with its usage in the mathematical setting.

We remark that this Zeta function enters in the study of the Riemann hypothesis in characteristic p. These are connected with the development of a suitable "Weil cohomology" theory in characteristic p which has coefficients valued in characteristic zero. For a review of the Weil conjectures, see e.g. [68]. We also note that this seems to fit with one of the "Atiyah fantasies" outlined in reference [69]. Here, our choice of cohomology theory is instead specified by a choice of nilpotent supercharge.

Our proposed relation between the supersymmetric index and the Zeta function also allows us to make sense of the Zeta function, even when the variety V is singular. This seems to line up with expectations from rigid cohomology.

That being said, there are some clear pitfalls compared with the case of characteristic zero. For example, a common strategy in the characteristic zero setting is to consider perturbations in the physical theory so as to localize the path integral sum around specific field configurations. Doing so in this setting can spoil the counting problem, since for example, the Zeta function of an elliptic curve depends quite sensitively on its arithmetic properties. Of course, the failure of the index to remain invariant under such perturbations is by itself a quite intriguing feature, and points to additional structure being present in the corresponding Hilbert space.

It would be interesting to develop this in detail, but we leave a full treatment for future work.

6 FI Parameters Revisited

Having sketched how to make sense of various field theories in characteristic p, we now turn to a potential physical application, in the context of large field ranges of a quantum field theory. The standard lore is that in a theory of quantum gravity, increasing the field range of a scalar leads to a breakdown in the low energy effective field theory. For super-Planckian field ranges, one does not expect semi-classical reasoning to carry over. In this section we revisit this class of questions from the perspective of reduction modulo p a prime number.

To keep things concrete, we focus on a 4D supersymmetric $U(1)$ gauge theory with a Fayet-Iliopoulos parameter. We have already sketched how to generate a supersymmetric quantum mechanics theory, as well as gauge theories in characteristic p, so we can already anticipate that the same algebraic manipulations which are used in characteristic zero will have characteristic p analogs. It was argued in [70] that 4D $\mathcal{N} = 1$ supergravity theories without a global R-symmetry are incompatible with the existence of an FI parameter. Indeed, the typical situation in a string compactification is that such "parameters" actually arise as vevs of background fields. Building on [6], references [7–9] argued that there is a potential loophole in such arguments if the FI parameter comes quantized in units of $2M_{\rm pl}^2$:

$$\xi = 2mM_{\rm pl}^2 \quad \text{for } m \in \mathbb{Z}. \tag{6.1}$$

where here $M_{\rm pl}$ refers to the reduced Planck mass, i.e. we have:

$$M_{\rm pl}^2 = \frac{1}{8\pi G_N}. \tag{6.2}$$

In all known string constructions, the resulting FI parameters appear to actually be "field dependent" that is, it is really just the background vev for another dynamical field. One could in principle imagine that such a large value of the FI parameter instead emerges from a suitably quantized flux. In Appendix D we present some evidence that this is indeed possible.

Here, we ask whether we can use our present perspective on field theory in characteristic p to study this and related questions where the field range becomes extremely large. We begin by writing down the bosonic sector of a 4D $\mathcal{N} = 1$ theory with gauge group $U(1)$ and chiral superfields $\Phi_1, ..., \Phi_n$ with charges $q_1, ..., q_n$. The condition that the spectrum is anomaly free means that $q_1 + ... + q_n = 0$. In characteristic zero, the bosonic sector of the Lagrangian contains the terms:

$$S \supset \int d^4x \left(-\frac{1}{4g^2} F_{\mu\upsilon} F^{\mu\nu} + \sum_i |\partial\Phi_i + q_i A\Phi_i|^2 - \frac{g^2}{2}\left(\sum_i q_i |\Phi_i|^2 - \xi \right)^2 \right), \tag{6.3}$$

where g refers to the gauge coupling of the $U(1)$ gauge theory. We now consider performing

a similar rescaling as that indicated in section 2. We assume that each field can move a minimal step $\Lambda_{\rm min}$. We also make the replacement $d^4x \mapsto \Lambda_{\rm max}^{-4}$ and $\partial \mapsto \Lambda_{\rm max}\partial$. Doing so, our proposal for a discretized action is:

$$S = \sum_{x\in X}\left(\frac{\Lambda_{\rm max}^2}{\Lambda_{\rm min}^2}\sum_i |D\varphi_i|^2 - \frac{g^2}{2}\frac{\Lambda_{\rm min}^4}{\Lambda_{\rm max}^4}\left(\sum_i q_i\,|\varphi_i|^2 - 2m\frac{M_{\rm pl}^2}{\Lambda_{\rm min}^2}\right)^2\right). \tag{6.4}$$

We make the assumption that the ratios of energies come in quantized steps so that we can set:

$$\frac{\Lambda_{\rm max}^2}{\Lambda_{\rm min}^2} = \frac{2\pi}{N}, \quad \frac{M_{\rm pl}^2}{\Lambda_{\rm min}^2} = M, \quad \frac{g^2}{2}\frac{\Lambda_{\rm min}^4}{\Lambda_{\rm max}^4} = \frac{2\pi}{N}B. \tag{6.5}$$

for some integers $B, K, N \in \mathbb{Z}$. The factors of π appearing here are actually rather natural. For example, we can also present these conditions as:

$$\frac{\Lambda_{\rm max}^2}{\Lambda_{\rm min}^2} = \frac{2\pi}{N}, \quad \frac{M_{\rm P}^2}{\Lambda_{\rm min}^2} = 8\pi K, \quad \alpha_{U(1)}\frac{\Lambda_{\rm min}^4}{\Lambda_{\rm max}^4} = \frac{B}{N}, \tag{6.6}$$

where $M_{\rm P}$ is the non-reduced Planck mass, i.e. $M_{\rm P}^2 = G_N^{-1}$ and $\alpha_{U(1)} = g^2/4\pi$. In any event, this motivates us to consider the discretized action:

$$S = \frac{2\pi}{p}\sum_{x\in X}\left(\sum_i |D\varphi_i|^2 - B\left(\sum_i q_i\,|\varphi_i|^2 - 2r\right)^2\right), \tag{6.7}$$

namely, we work modulo $N = p$ a prime number, and we have introduced an integer parameter $r = mK$. Here, we have also assumed that our fields are valued in $\mathbb{F}_p(\widehat{i})$, and the $|\cdot|$ notation refers to expanding out as a square, i.e.:

$$|\varphi|^2 = a^2 - \widehat{i}^2 b^2 \quad \text{for} \quad \varphi = a + \widehat{i}b \quad \text{with } a, b \in \mathbb{F}_p \tag{6.8}$$

In this case, we can apply all the machinery previously developed. One immediate observation is that in working mod p, it could happen that the discretized analog of the FI parameter now vanishes. So, an expansion around $\xi = 0$ and a super-Planckian FI parameter can in such cases appear quite similar in characteristic p.

Let us now turn to the vacua of the system. In characteristic zero we label these as zeros of the effective potential, modulo $U(1)$ gauge transformations. This defines a toric variety Y and the procedure just outlined specifies a symplectic quotient:

$$Y = (\mathbb{C}^*)^n\,//U(1). \tag{6.9}$$

One can generalize this in various ways by including additional fields, as well as multiple $U(1)$ factors. Note that because we demanded that the q_i's sum to zero, we have a toric

Calabi-Yau space. If we had considered the analogous problem in a 2D system, we could relax these conditions further. For additional discussion of the 2D field theory analysis, see e.g. [71].

Two canonical examples of non-compact Calabi-Yau spaces which are captured by such a symplectic quotient include $\mathcal{O}(-n) \to \mathbb{CP}^{n-1}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. With suitable charge assignments, a positive value of the FI parameter specifies the volume of the compact $\mathbb{CP}^{n-1}$ and $\mathbb{CP}^1$ factor. In the case of the conifold, switching to negative values of the FI parameter signals a flop transition.

Consider next the related analysis in characteristic p. See for example [72] for some discussion of toric geometry in characteristic p. In this case, we can still define an appropriate quotient by a group action, but now it is of the form:

$$Y = \left(\mathbb{F}_p(\widehat{i})\right)^n //U(1, \mathbb{F}_p(\widehat{i})). \tag{6.10}$$

As an illustrative example, in the case of $\mathcal{O}(-n) \to \mathbb{P}^{n-1}$, the D-term constraint can be written as:

$$|\varphi_1|^2 + |\varphi_2|^2 + ... + |\varphi_n|^2 - n\,|z|^2 = 2r. \tag{6.11}$$

with fields φ_i of charge $+1$ and z of charge $-n$. Suppose we fix the value of z. We note that this specifies a $\mathbb{P}^{n-1}$ over the base field $\mathbb{F}_p(\widehat{i})$. Indeed, given $\Phi_i \in \mathbb{F}_p(\widehat{i})$, we can consider the identification of homogeneous coordinates $[\Phi_1, \Phi_2, ..., \Phi_n] \sim [\lambda\Phi_1, \lambda\Phi_2, ..., \lambda\Phi_n]$ for $\lambda \in \mathbb{F}_p(\widehat{i})$, and this defines a $\mathbb{P}^{n-1}$. Given λ, we can write such an element as:

$$\lambda = c \cdot u \tag{6.12}$$

for $c \in \mathbb{F}_p$ and $u \in U(1, \mathbb{F}_p(\widehat{i}))$. So, we expect that even in characteristic p, we can still identify this symplectic quotient with projective space. An interesting feature of this analysis is that provided $2r + n\,|z|^2$ is non-zero, the number of points in this $\mathbb{P}^{n-1}$ is always the same, even if we vary the FI parameter. To see this, we note that our $\mathbb{P}^{n-1}$ can instead be written as a coset space:

$$\mathbb{P}^{n-1} = \frac{SU(n, \mathbb{F}_p(\widehat{i}))}{SU(n-1, \mathbb{F}_p(\widehat{i})) \times U(1, \mathbb{F}_p(\widehat{i}))}, \tag{6.13}$$

and so given a single point in this space, we can use the transitive $SU(n, \mathbb{F}_p(\widehat{i}))$ group action to reach any other point.

We can also count the number of such points. To do this, suppose that in the presentation in terms of the homogeneous coordinates $[\Phi_1, \Phi_2, ..., \Phi_n]$ the coordinate Φ_1 is non-zero. Then, we get an affine patch with coordinates Φ_i/Φ_1 for $i = 2, ..., n$. There are $n-1$ such coordinates, so we get a total of p^{n-1} distinct points. Next, suppose that $\Phi_1 = 0$ but that Φ_2 is non-zero. In this patch, we have coordinates Φ_i/Φ_2 for $i = 3, ..., n$, and we get a total of p^{n-2} distinct points. Continuing in this way, we can get all the way down to all $\Phi_i = 0$

for $i = 1, ..., n-1$, and we are left with the single point $[0, ..., 1]$, which counts as just one point. The total number of points is then:

$$\left|\mathbb{P}^{n-1}\right| = p^{n-1} + p^{n-2} + ... + p + 1 = \frac{1-p^n}{1-p}. \tag{6.14}$$

We note that this is essentially a "motivic" decomposition of $\mathbb{P}^{n-1}$ as:

$$\mathbb{P}^{n-1} = \mathbb{A}^{n-1} \oplus \mathbb{A}^{n-2} \oplus ... \oplus \mathbb{A}^1 \oplus \mathbb{A}^0. \tag{6.15}$$

More generally, consider varying the value of $r' = 2r + n|z|^2$. The D-term constraint can then be presented as:

$$|\varphi_1|^2 + |\varphi_2|^2 + ... + |\varphi_n|^2 = r', \tag{6.16}$$

modulo $U(1, \mathbb{F}_p(\widehat{i}))$ transformations. If there were no symplectic quotient, we would just get a copy of $\mathbb{A}^n$, which has p^n points. Varying r' over $\mathbb{F}_p$, we see that there are $(p-1)$ non-zero values, and one where it vanishes. So, the total number of points in $\mathbb{A}^n$ can be written as:

$$|\mathbb{A}^n| = (p-1)\left|\mathbb{P}^{n-1}\right| + \left|\mathbb{P}^{n-1}_{r'=0}\right|, \tag{6.17}$$

so we also learn that:

$$\left|\mathbb{P}^{n-1}_{r'=0}\right| = p^n - (p-1)\frac{1-p^n}{1-p} = 1. \tag{6.18}$$

Consequently, if we now vary z, we see that we can visualize the total space as a collection of $\mathbb{P}^{n-1}$ spaces. These split into two types, ones where $n|z|^2 + 2r \neq 0$, and those with $n|z|^2 + 2r = 0$. We note that both can occur "frequently" when in characteristic p. Another comment is that even as we vary the FI parameter, we do not recover a "macroscopic geometry." Indeed, each of our shells has a finite number of points.

As another example, consider the case of a conifold, namely a quadric. From the perspective of our $U(1)$ gauge theory, we introduce two charge $+1$ fields u_1, u_2, and two charge -1 fields v_1 and v_2. The point set is then captured by the condition:

$$|u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 = 2r. \tag{6.19}$$

Here, we observe another curiosity: In characteristic p, the notion of r "positive and negative" does not really make sense. Of course, the action $r \to -r$ does still switch the roles of the u and v coordinates, corresponding to a flop transition, but we can no longer identify this with just continuing a Kähler class to negative values.

7 Geometric Engineering in Characteristic p

As already mentioned in the Introduction, one of the motivations for the present work is that some of the main tools used in constructing string vacua actually make use of techniques from algebraic geometry, with little explicit use made of the actual spacetime metric in the extra dimensions. This is usually treated as problematic, because it means that many non-holomorphic quantities of interest such as the masses of particles can at best be obtained in some approximation scheme. On the other hand, the very fact that these constructions are often formulated in the algebraic setting is a *welcome* feature in studying the passage to characteristic p. Based on our physical considerations presented in section 2, we view the resulting arithmetic geometries as the highly quantum regime of a string compactification. This by itself is a rather appealing discretization of a string compactification, and seems motivation enough. Here we consider a variant of geometric engineering [19–22] but in characteristic p.

Our plan in this section will be to make use of some of the more rigorously established aspects of geometric engineering in characteristic zero, now transported to the characteristic $p > 0$ setting. Our string compactification geometries will be Calabi-Yau varieties in characteristic p. This means the canonical sheaf is trivial. We will be interested in geometric engineering, the framework used to connect certain singular string compactification geometries to partially twisted field theories. For a recent overview of geometric engineering in characteristic zero, we refer the interested reader to reference [73]. For some recent discussion of Calabi-Yau spaces over finite fields, see e.g. [30, 31, 33].

The first non-trivial example we wish to consider involves a correspondence between the Hitchin system on a genus g complex curve Σ with an ADE gauge group G, and a local singular Calabi-Yau threefold Y comprised of a curve Σ of ADE singularities.

For example, in the case of an A_{M-1} singularity, the singularity can be presented as the hypersurface equation:

$$xy = z^M, \tag{7.1}$$

where $(x = y = z = 0)$ denotes the location of the curve. We will denote by Y_t the smoothings of the threefold. The physics of this system has been investigated in a number of papers, for a partial list of examples see e.g. [21, 74–79]. Let us briefly review the match between the two moduli spaces, working at smooth points.

The correspondence involves matching the Hitchin moduli space to the moduli spaced defined by the Weil intermediate Jacobian of the Calabi-Yau. In this correspondence, the base of the Hitchin moduli space defined by Casimir invariants of a Higgs field maps to smoothing deformations of the Calabi-Yau. The data of holonomies in the Hitchin system maps to periods of a three-form potential defined on the Calabi-Yau threefold. On the Hitchin system side of the story, we specify a pair (E, Φ) consisting of a principal G bundle

$\mathcal{E}$ and a Higgs field Φ, which defines a map:

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}_\Sigma, \tag{7.2}$$

with a suitable notion of stability for the Higgs bundle. The first important match between these two structures is the mapping between coefficients in the spectral equation of the Higgs field (viewed as a hypersurface in the canonical bundle over Σ) and smoothing deformations of the local Calabi-Yau threefold. In more detail, recall that the spectral equation for the Higgs field in the fundamental representation is:

$$\det(u\mathbb{I}_{M\times M} - \Phi) = \sum_{i=0}^{M} c_i u^{M-i} = 0, \tag{7.3}$$

with u a section of the canonical bundle, and c_i a Casimir invariant built from the Higgs field, which we view as a section of $(\mathcal{K}_\Sigma)^{\otimes M}$. These map to unfoldings of the singularity:

$$xy = \sum_{i=0}^{M} c_i u^{M-i}. \tag{7.4}$$

The zero set of the spectral equation specifies a spectral cover of the original curve:

$$\widetilde{\Sigma} \stackrel{\pi}{\to} \Sigma. \tag{7.5}$$

Additionally, we can equip $\widetilde{\Sigma}$ with a line bundle $\widetilde{\mathcal{L}}$, and via the spectral cover construction [80], the push-forward map under π_* generates a vector bundle. This line bundle can also be viewed as being specified by a point in the Jacobian of $\widetilde{\Sigma}$, and this in turn has a direct analog in the smoothed Calabi-Yau threefold geometry Y_t as a point in the (Weil) intermediate Jacobian $\mathcal{J}(Y_t)$.

An important feature of establishing this correspondence rigorously is that it can actually be formulated *algebraically*, with no direct reference to metric data.[8] Given everything we have seen so far, it would seem natural to expect a correspondence over characteristic p to also hold. Again working with respect to the A-type case, we expect that on the Hitchin system side of the correspondence will involve an $SL(N, \mathbb{F}_q)$ vector bundle E, and a Higgs field

$$\Phi : E \to E \otimes K_\Sigma. \tag{7.6}$$

Encouragingly, we note that some work has been done on developing Hitchin system in characteristic p, including in the proof of the Fundamental Lemma of the Langlands program (see e.g. [81–83]). A question which should be addressed in this direction would center on the effects of changing the base field. Presumably, the closest analog to the characterstic zero correspondence holds for the case of $\overline{\mathbb{F}_p}$, but we expect that the more general situation

[8] We thank R. Donagi and T. Pantev for discussions on this point.

over a finite field is also well-defined.

There are various generalizations of this basic correspondence. In characteristic zero, it is also expected that we can instead consider a Kähler manifold S of dimension d equipped with a Higgs bundle with structure group G an ADE group specified by the pair $(\mathcal{E}, \Phi)$. In this case, the expectation is that there is again a correspondence, but this time with a local Calabi-Yau $(d+2)$-fold Y specified by a Kähler manifold of ADE singularities (see e.g. [74, 84–86]. A non-trivial feature of the $d > 1$ case is the appearance of non-zero "bulk fluxes" on the Calabi-Yau side, which in turn is expected to be specified (at a suitable smoothing of the singular Calabi-Yau) by the Deligne-Beilinson cohomology [87, 75]. While even this has not been established in full generality, one expects that an algebraic correspondence will also be available in this case as well.

Indeed, we observe that part of this correspondence is straightforward to establish, both in characteristic zero and in characteristic p. For ease of exposition, we assume that the canonical bundle of S is very ample. In this case, we can construct the spectral equation in the total space of the canonical bundle for S, and match the corresponding Casimir invariants with smoothing deformations of the Calabi-Yau Y. In all these cases, we expect that a suitable analog of this correspondence to work for meromorphic Higgs fields with singularities specified along various subspaces. As a particular case of interest, observe that we can now specify a characteristic p version of the Vafa-Witten system [88] on a Kähler surface. This in turn suggests a potential way to connect with the GL twist of reference [89], though in the geometric engineering setting, this is usually not phrased as a purely holomorphic problem.[9] For example, recall that in characteristic zero, we can engineer $\mathcal{N} = 4$ Super Yang-Mills theory by working with type IIB strings on a $\mathbb{E} \times \mathbb{C}^2/\Gamma_{ADE}$, with $\mathbb{E}$ an elliptic curve and $\mathbb{C}^2/\Gamma_{ADE}$ an ADE singularity, as defined by a singular hypersurface equation. This sort of geometry still makes sense in characteristic p, so presumably we can use this to set up a characteristic p analog of reference [89].

[9] Rather, the local model in question involves branes wrapped on the local Calabi-Yau fourfold T^*M_4 with M_4 a four-manifold. It would clearly be interesting to determine whether a characteristic p version of this story makes sense, particularly sense the GL twist figures prominently in the geometric Langlands program [89].

8 More General Numbers

In the previous sections we showed that there is a rich geometric structure present in the special case where we quantize the reduced Planck constant in units of p a prime. Here, we turn to the more general situation as specified by working with:

$$\hbar = \frac{N}{2\pi}, \tag{8.1}$$

where now we assume N is not a prime number, and instead factorizes as:

$$N = p_1^{a_1}...p_k^{a_k}. \tag{8.2}$$

Our plan will be to first show how to extend our considerations when there is a single prime power appearing, and especially the emergence of topological structures in the large N limit.

Consider first the special case where $N = p^a$. Since our starting point was a basis of fields with integer coefficients, we are free to consider a p-adic expansion for any such integer of the form:

$$\phi = \sum_i \phi_i p^i. \tag{8.3}$$

Working in terms of polynomials in $\mathbb{Z}_p[t_1, ..., t_D]$, with $\mathbb{Z}_p$ the ring of p-adic integers, we see that we can also construct an action, and perform a similar p-adic expansion:

$$S = \sum_{j \geq 0} S_j p^j, \tag{8.4}$$

which truncates at finite order (for a given field configuration). Reduction modulo N means that we simply drop the higher order terms in this expansion.

This p-adic expansion also shows that at least in the limit $a \to \infty$, there is a natural topology for our basis of fields and our action. To see how it comes about, we observe first that for each $n \in \mathbb{N}$, the space $\mathbb{Z}/p^n\mathbb{Z}$ can be equipped with the discrete topology (each point is both open and closed). Next, we can view the p-adic integers $\mathbb{Z}_p$ as obtained from the inverse limit (see Appendix F):

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}. \tag{8.5}$$

Consequently, we can equip $\mathbb{Z}_p$ with the relative product topology. This turns out to generate the same topology as we would get if we had just introduced the p-adic norm $|\cdot|_p$ from the start. For an integer $\phi = p^n\phi'$ with ϕ' relatively prime to p, we have:

$$|\phi|_p = p^{-n}, \tag{8.6}$$

So, higher order powers in p are actually small corrections! Building the field of fractions

out of $\mathbb{Z}_p$, we reach the p-adic numbers $\mathbb{Q}_p$, and the p-adic norm extends in the expected way (e.g. $|p^n|_p = p^{-n}$). Clearly, this is a different notion of proximity from what one is accustomed to dealing with in using the real numbers, but if all we consider is the integers, there is a priori no issue with introducing such a number system.

The specification of the coefficients ϕ_j and S_j in equations (8.3) and (8.4) is actually somewhat subtle. One's first inclination might to be just fix the coefficients according to coefficients valued in $\{0, ..., p-1\}$, with each of these having a clear interpretation in $\mathbb{F}_p$. This turns out to only work for the leading order coefficients. The main issue is that we would like to have a suitable notion of coefficient-wise addition and multiplication so that we need not worry about "carry over" from arithmetic operations.

In fact, this precise issue can be dealt with by instead treating our physical fields and our action as Witt vectors, something we review in Appendix G. The key point is that for components of two Witt vectors U and V, we do have a natural mod p^{i+1} relation of the form:

$$(U+V)_i = U_i + V_i \tag{8.7}$$

$$(UV)_i = U_i V_i. \tag{8.8}$$

For our purposes, we mainly need to apply this formalism in the case of the p-adic integers $\mathbb{Z}_p$. In that setting we can present each field ϕ as well as the action in terms of its Teichmüller representative. In this more abstract setting, we can now work with schemes defined over $\mathbb{Z}_p$ reduced modulo p^a.

With this in place, we can now set up a very similar line of development to what we initially considered in the case of physics over finite fields of characteristic p. In this case, we can choose to consider bosonic physical fields as locally specified by polynomials in $\mathbb{Z}_p[t_1, ..., t_D]$. All of the geometric flavor introduced previously still appears to make sense, provided we interpret our geometric structures as varieties over the base field $\mathbb{Q}_p$ reduced modulo p^a, or even better, as schemes over $\mathbb{Z}_p$ reduced modulo p^a. This latter feature also illustrates that restricting to just polynomials with $\mathbb{Z}_p$ coefficients should provide a suitable notion of "convergence" of these power series. Note that there is also a natural boundary which emerges in these geometries, since $\mathbb{Z}_p$ consists of elements with p-adic norm less than or equal to one (i.e. it specifies a disk).

Indeed, a helpful feature of working over the p-adics is that we can then also speak of an emergent topology in the large a limit. In the case of characteristic p varieties, we already saw hints of this emergent structure in our discussion of crystalline cohomology. Here, we see it appearing again, albeit in a somewhat different guise. Let us also note that this also provides a more refined topology than both the Zariski topology and the étale topology that are implicit in our earlier treatment of characteristic p spaces. Returning to our very brief discussion of symmetric bilinear forms defined on $T^*X \otimes T^*X$, we see that the corresponding

p-adic expansion: [10]

$$h_{\mu\nu} = \sum_i h^{(i)}_{\mu\nu} p^i, \tag{8.9}$$

also means that there is a suitable notion of a metric for such schemes.

As an amusing application, consider points in the 2D "spacetime" obtained from $\mathbb{Q}_p \times \mathbb{Q}_p$ with $h_{\mu\nu}$ specifying the standard symmetric bilinear form of Minkowski space, namely we can set:

$$h_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2. \tag{8.10}$$

If we restrict to coordinates valued in the rational numbers, we can speak of timelike values $h_{\mu\nu} x^\mu x^\nu > 0$ and spacelike values $h_{\mu\nu} x^\mu x^\nu$, as well as the lightcone $h_{\mu\nu} x^\mu x^\nu = 0$. In the extension to $\mathbb{Q}_p$ where there is no complete ordering, only the notion of the lightcone persists. Another comment here is that in the reduction modulo p^a, we see that a lightcone can take the form:

$$x^0 = x^1 + \alpha p^a, \tag{8.11}$$

which makes sense in any characteristic. So in other words, a single lightcone defined in characteristic p^a breaks up into several disjoint lines inside of $\mathbb{Q} \times \mathbb{Q}$. Consider the large a limit. In the real topology these lines get further and further away from one another, but in the p-adic topology these lines get closer and closer together.

Of course, the p-adic numbers are also rather far removed from our usual notions of metric and distance, at least as far as they are applied in many continuum physical problems. We now argue that in the large p^a limit, this sort of structure also naturally appears. To see why, we note that in evaluating our correlation functions, we actually make implicit reference to the metric on $\mathbb{C}^*$. This follows simply from the fact that our action principle is really formulated in terms of additive characters of the given ring, namely we have:

$$\exp\left(\frac{2\pi i}{N} S\right) \in S^1 \subset \mathbb{C}^*. \tag{8.12}$$

Convergence with respect to the metric on $\mathbb{C}^*$ is of course a very different notion from that specified by the p-adic norm, and gives rise to a completely different sort of topology and notion of "proximity." In the context of our quantum theory, however, we see that actual probabilities / expectation values as computed by the path integral still implicitly make reference to this real metric structure. This raises an important subtlety: if we blindly take the entire infinite series in the p-adic expansion, then we will often produce numbers in $\mathbb{Z}_p$ which are no longer integers. Even so, the notion of a map to $\mathbb{C}^*$ still makes sense because when we reduce modulo p^a, we get back an integer and the corresponding character is well-defined.

Evaluating on a given physical field configuration, the action can still converge in either

[10] Compared with earlier, we now label the spacetime indices by Greek rather than Latin indices since here a refers to the exponent of a prime power.

topology, it just depends on how we take the large N limit. One might argue that for evaluating correlation functions the actual quantity of interest is:

$$\frac{1}{p^a}S = S_a + \frac{1}{p}S_{a-1} + ... + \frac{1}{p^a}S_0, \tag{8.13}$$

with the leading term set by S_a. In this case, convergence is best thought of in terms of the usual real numbers. We can pass between these two expansions through the formal mapping:

$$\mathbb{Q}_p \leftrightarrow \mathbb{R} \tag{8.14}$$

$$\hbar = \frac{p^a}{2\pi} \leftrightarrow \hbar = 1 \tag{8.15}$$

$$S_j \leftrightarrow S_{a-j}\,. \tag{8.16}$$

To get a character map which converges in $\mathbb{C}^*$, we need to truncate the p-adic expansion so that $p^{-a}S$ remains small in the real topology. This means that for a fixed value of $N = p^a$ we would need to truncate to terms of degree p^{a-1} or lower:

$$\phi \sim \sum_{j=0}^{a-1}\phi_j p^j \;\text{ and }\; S \sim \sum_{j=0}^{a-1}S_j p^j. \tag{8.17}$$

We take this to mean that as we pass to the extremely quantum regime where $\hbar \to \infty$, we actually recover a semblance of standard quantum fields. Note also that the "classical limit" corresponds to holding the expansion degree fixed at some maximal $j_{\max}$ and sending $a \to \infty$.

So, depending on how we take our large N limit, we can approach either the p-adic or real topology. If we take the limit at the level of the action, then we pass to the p-adic topology whereas if we take the limit in the space of characters valued in $\mathbb{C}^*$, then we instead pass to the real topology. In the latter case, we use the truncated p-adic expansion of equation (8.9) also provides us with a metric on our spacetime. Let us note that in the lattice approximation discussed in section 2 as well as the worked example in Appendix A, there is a sense in which we are still referencing the standard real topology by performing "nearest neighbor differences." In such situations, the passage back to the continuum is the standard one. In the more abstract setting based on polynomials, more care as warranted, but the above procedure shows how to accommodate this situation as well.

Consider next the case where N has distinct prime factors. To study this case, we continue with our theme of interpreting physical structures in the language of arithmetic geometry. We begin by considering the special case:

$$N = p_1...p_m, \tag{8.18}$$

namely the power of any prime factor is at most one. Thankfully, the relevant structures

have also been developed by mathematicians. The main idea we want to exploit is to view the integers $\mathbb{Z}$ as a coordinate ring for a "curve".[11] In algebraic geometry, we construct our geometric spaces by dealing with the spectrum of the ring, namely the set of maximal prime ideals. In the present context, this is just the ideals generated by the primes as well as 0:

$$\operatorname{Spec}\mathbb{Z} = \{\langle p\rangle \text{ for } p \text{ a prime}\} \cup \{0\}. \tag{8.19}$$

In this way of thinking, $\operatorname{Spec}\mathbb{Z}$ is just an affine curve with points specified by these maximal ideals. The meaning of an integer such as $N = p_1^{a_1}...p_k^{a_k}$ is then that it is a collection of k points. Moreover, the exponent a_i indicates the "thickness" of that point as a non-reduced scheme. See Appendix H for a brief discussion on some geometric aspects $\operatorname{Spec}\mathbb{Z}$.

For a given physical field configuration, then, we can view our action S as a function over the curve $\operatorname{Spec}\mathbb{Z}^\times$, namely, the affine line with the origin deleted. We can, of course, consider localization near any prime factor of N, and this will lead us to a power series expansion in that prime. We denote the resulting subscheme of $\operatorname{Spec}\mathbb{Z}$ as $(\operatorname{Spec}\mathbb{Z})_N$.

With this in mind, we considering fibering all of the construction developed previously for a fixed prime p,[12] to construct a larger spacetime and target space:

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{X} \\ & & \downarrow \\ & & \operatorname{Spec}\mathbb{Z}^\times \end{array}, \quad \text{and} \quad \begin{array}{ccc} Y & \longrightarrow & \widetilde{Y} \\ & & \downarrow \\ & & \operatorname{Spec}\mathbb{Z}^\times \end{array}. \tag{8.20}$$

Each stalk of the fiber is meant to be interpreted as a variety in characteristic p. We can further supplement this by working over different finite fields such as $\mathbb{F}_q$.

Now, extending the spacetime and target space in this way, we see that there is a sense in which it is actually more naturally to perform a path integral over all maps from $\widetilde{X}$ to $\widetilde{Y}$. In terms of the geometry of $\operatorname{Spec}\mathbb{Z}$, this means that we instead take a product over all possible prime numbers, but also all maps between these primes by treating $\hbar$ as a map:

$$2\pi\hbar : \operatorname{Spec}\mathbb{Z}^\times \to \operatorname{Spec}\mathbb{Z}^\times, \tag{8.21}$$

where we assume $2\pi\hbar(0) = 0$. This removes the restrictions on exponents introduced in equation (8.18).

A path integral phase factor which includes these effects is given by:

$$\prod_{\hbar}\prod_{p}\prod_{x\in X_p} e^{\left(\frac{2\pi i}{\hbar(p)}S_x\right)} e^{2\pi i S_{\text{extra}}[\hbar,x]} \simeq \prod_{x\in X_N} e^{\left(\frac{2\pi i}{N}S_x^{\text{eff}}\right)}, \tag{8.22}$$

[11] Typically, one makes sense of curves over a field. Here, we are relaxing even this condition, so the geometric picture will suffer somewhat. We will nevertheless persist with this language since it is helpful.

[12] We thank R. Donagi for emphasizing this feature of reduction modulo p to us.

where in the above, the additional term S_{extra} also captures more general possible hopping terms between primes. On the righthand side of this expression we view the appearance of a fixed N as the result of performing this product, with an effective action S^{eff} encapsulating these effects. Here, X_N simply refers to the restriction of the fibers of $\widetilde{X}$ to the subscheme $(\operatorname{Spec}\mathbb{Z})_N$. To get a good approximation of S, we can also adopt a Wilsonian perspective, integrating over primes p, starting with the small ones, and then moving to the bigger ones. This provides a sense in which we can pass from short distances back to long distances.

Operator correlation functions also generalize. We view a physical field $\phi(p, x_p)$ as having support over the total space $\widetilde{X}$ given as the fibration of X over $\operatorname{Spec}\mathbb{Z}^{\times}$. For each prime factor, the definition of the operator is specified in exactly the same way, the only issue is that we should now write a local operator such as the one of line (2.15) as:

$$U(p, x_p) = \exp\left(\frac{i}{\hbar(p)}\phi(x_p)\right), \tag{8.23}$$

with $p \in \operatorname{Spec}\mathbb{Z}^{\times}$ and $x_p \in X_p$.

Clearly, evaluating correlation functions with this sort of procedure can quickly become unwieldy. A well-motivated approximation is obtained by restricting $2\pi\hbar$ to a special class of maps of the form:

$$2\pi\hbar : \operatorname{Spec}\mathbb{Z}^{\times} \to \operatorname{Spec}\mathbb{Z}^{\times} \tag{8.24}$$

$$x \mapsto x^n. \tag{8.25}$$

The reason these maps are "special" is that they simply send an ordinary point to a "fat point" of the same type, just increasing its multiplicity. This affords us a notion of locality on the primes, so it seems reasonable to make this further restriction.

At a practical level this also makes the reduction over a given prime more tractable, but still quite flexible. In this case, the path integral phase factor collapses to:

$$\prod_{n\in\mathbb{N}}\prod_{p}\prod_{x\in X_p} e^{\left(\frac{2\pi i}{p^n}S_x\right)}. \tag{8.26}$$

Additionally, the structure of operators such as those of line (8.23) also simplifies, and the reduction mod p^n can now be applied fiberwise.

All of the remarks about the emergence of a p-adic and real topology also apply to this enlarged setting. Assuming there is an approximation of the full path integral in terms of an effective action and some integer $N = p_1^{a_1}...p_k^{a_k}$, we can consider each prime factor individually and then the large N limit amounts to assuming that the exponents a_i are all sufficiently large. We thus expect this to produce the expected real topological structure in the continuum limit.

8.1 Zeta Functions Revisited

As we have already remarked, evaluating the full path integral of line (8.22) is somewhat unwieldy, but at least provides a general framework for recovering continuum notions of spacetime. Now, in the context of applications to number theory, the idea of starting with a general algebraic variety over $\mathbb{Q}$ and recasting this as a scheme over $\mathbb{Z}$ is a well known procedure. In this context, reduction modulo a prime p then provides important arithmetic information on the behavior of the system. For a generic algebraic variety, reduction with respect to a generic prime p will produce a non-singular variety over $\mathbb{F}_p$, and in such cases we can speak of the corresponding Zeta function $Z_{V,p}(z)$, and its relation to a supersymmetric index:

$$\sum_{n\geq 1} \mathrm{Tr}_n \left((-1)^{\mathbf{F}} z^n\right) = \log Z_{V,p}(z) = \sum_{n\geq 1} \#V(\mathbb{F}_{q^n}) \frac{z^n}{n}. \tag{8.27}$$

We have also argued that this quantity can also be interpreted as the supersymmetric index of a suitable characteristic p quantum mechanics problem.

Reduction modulo p may also result in a singular space, and this forms the basis for defining the theory of the "conductor" of a variety. For generic varieties this sort of singular reduction happens for a finite number of primes. We have also seen in subsection 5.3 that on physical grounds, the structure of the supersymmetric index should allow us to make sense of the Zeta function, even if there is a singular reduction.

Now, in the more general setting just introduced, we have been considering a further fibration over $\mathrm{Spec}\,\mathbb{Z}^\times$, so we can speak of the supersymmetric index obtained from working with all the different primes. If we restrict to the special case where the map $\hbar/2\pi$ of line (8.24) is just the identity map (no fat points at all) then we just take the product over all the different zeta functions. This produces the expression:

$$Z_{V,\mathbb{Q}}(\{z_p\}) \equiv \prod_p Z_{V,p}(z_p). \tag{8.28}$$

In the arithmetic geometry literature it is customary to work in terms of a single uniform fugacity. Introducing a complex number s (with suitable domain of definition to ensure convergence of the product), we can write $z_p = p^{-s}$. Note that this means larger primes are "penalized" in the associated partition function. Making this substitution, we arrive at the Zeta function:

$$\zeta_{V,\mathbb{Q}}(s) \equiv \prod_p \zeta_{V,p}(s), \tag{8.29}$$

where we changed notation ($\zeta = Z$) to emphasize the different variable dependence.

The present formulation also suggests some natural generalization of these sorts of formulae. Instead of just dealing with the identity map for $2\pi\hbar : \mathrm{Spec}\,\mathbb{Z}^\times \to \mathrm{Spec}\,\mathbb{Z}^\times$, we can consider more general powers. Evaluating the corresponding local Zeta functions and taking the product over prime powers provides a more general class of possible Zeta functions.

8.2 Geometric Engineering Revisited

It is also interesting to revisit our discussion of geometric engineering, especially as a way to formulate a suitable notion of gauge theory on the bigger space $\widetilde{X} \to \text{Spec } \mathbb{Z}^{\times}$, and thus as a way to formulate mathematical (and physical!) quantities of interest.

The general geometric notions we need are specified by working with arithmetic schemes. The idea is to fix some variety V over an algebraic number field K (i.e. some finite field extension of the rational numbers $\mathbb{Q}$), and consider the ring of integers $\mathcal{O}_K$. Then, we can consider fibration $V \to$Spec$(\mathcal{O}_K)$ as obtained by reduction of the variety at prime ideals of $\mathcal{O}_K$. Doing so, the total space has "one dimension more" than the reduction. Clearly, this is precisely the situation outlined previously in the special case where $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. A simplified but still important example is provided by the case of a algebraic curves over K. In this case, the total space is referred to as an arithmetic surface S. See figure 4 for a depiction.

Our aim in this brief subsection will be to sketch how to use notions of geometric engineering to sketch a formulation of a gauge theory on an arithmetic surface S. We expect that similar notions also work for more general arithmetic varieties since geometric engineering also extends to this broader setting. We will also find it convenient to allow K to sometimes differ from $\mathbb{Q}$, even though we expect the physically most interesting case is likely provided by the "simplest situation." One reason for doing this is that in previous discussion of geometric engineering in characteristic p, we saw that the sharpest analogy with the characteristic zero case suggests working with the algebraic closure $\overline{\mathbb{F}}_p$.

To formulate gauge theory on S, we view it as an arithmetic surface of ADE singularities X such that the total space is Calabi-Yau. We note that such a notion makes sense because even in the setting of an arithmetic variety defined over a ring of integers such as $\mathcal{O}_K$, we can still speak of a canonical sheaf (see reference [90]). To be explicit, consider the special case of an A-type singularity. Then, we would write:

$$y^2 = x^2 + u^N, \tag{8.30}$$

where the locus $x = y = z = 0$ defines S. Of course, in the "standard setting," we would view this as engineering the Vafa-Witten system [88] as familiar from model building in F-theory [74,91]. The non-trivial step in the characteristic zero setting has to do with ensuring that the two notions of moduli spaces from gauge theory and singular Calabi-Yau geometry actual specify the same degrees of freedom.

Here, we will simply use the geometry as a way to *define* what we could possibly mean by gauge theory on the arithmetic surface, leaving a complete treatment for future work. In accord with the usual characteristic zero case, we expect to have a notion of a vector bundle $\mathcal{E}$ with an A-type structure group $SL(N, K)$, and a Higgs field. Now, to see the appearance of the vector bundle in our setting, we observe that we can perform blowups of X, even in

the arithmetic setting. Doing so, we observe the appearance of "fibral divisors" $D_1, ..., D_N$. These can be viewed as divisors of an ADE singularity, which is then further fibered over S.[13] We can now see the appearance of the 2-cycle class $[S]$ in the intersection pairing:

$$D_i \cdot D_j = C_{ij}[S], \tag{8.31}$$

where C_{ij} with $i, j = 1, ..., N$ is the usual intersection theoretic Cartan matrix (-2's on the diagonal). Taking linear combinations of effective divisors, we then build up the usual notion of a root system fibered over S. Implicitly, then, the deformations of line (8.30) are specifying the Casimir invariants of a Higgs field:

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}_S, \tag{8.32}$$

where the notion of $\mathcal{K}_S$ as the canonical sheaf makes sense for an arithmetic surface, and we view $\mathcal{E}$ as a sheaf on S which admits a group action by $SL(N, K)$. Now, for each prime in $p \in O_K$, we can fix our attention on the corresponding stalk S_p. This is akin to the characteristic p Hitchin system we already encountered.

An interesting feature of this setup is that we can now discuss surface operators, as associated with specifying a prescribed singularity structure for our Higgs field along a curve (or curves) in S. In the context of geometric engineering, this has been analyzed for example in references [74–76]. Call one such curve Σ. Then, this also specifying a divisor in S, and the Higgs field has a residue along Σ which is some element in the Lie algebra $\mathfrak{sl}(N, K)$:

$$\mathrm{Res}_C \Phi \in \mathfrak{sl}(N, K). \tag{8.33}$$

In the geometric engineering setting, we can generate such profiles by colliding different singularities together. For example, still within the context of our setup as in line (8.30), introduce another arithmetic surface S' locally specified as the hypersurface $x = y = v = 0$. Then, we can consider:

$$y^2 = x^2 + u^N v^M, \tag{8.34}$$

as associated with $SL(N, K)$ gauge theory on $u = 0$ and $SL(M, K)$ gauge theory on $v = 0$. The surfaces S and S' intersect along $u = v = 0$ which we view as a horizontal Arakelov divisor in S.

In the characteristic zero setting we would say that there are "matter fields on Σ" as specified by elements of $H^0(\Sigma, \mathcal{K}_\Sigma^{1/2} \otimes (\mathcal{E} \otimes \mathcal{E}')\,|_C)$. Presumably there is a suitable concept of this for divisors in arithmetic surfaces, via the analog of theta functions. In the special case where the bulk vector bundles are trivial, there exists a Lie algebra valued pairing for the zero modes (see [74] for the precise definitions) which we can use to specify the value of

[13] Here, the proper notion of divisor, and intersection of divisors implicitly makes reference to Arakelov's intersection theory [92, 93] (see reference [90] for an introduction). This is necessary to stipulate because we need to be able to specify what happens "at infinity," in Spec $\mathcal{O}_K$.

the Higgs field residue:

$$\mathrm{Res}_C \Phi = \langle \psi^c, \psi \rangle \, . \tag{8.35}$$

So, using concepts from geometric engineering, notions of surface operators (at least for Higgs fields) and gauge theory over S still appear to make sense.

It would also be natural to investigate the effects of S-duality on such gauge theories. Taking our cue from the characteristic zero setting, we can consider a product of an elliptic curve $\mathbb{E}$ and an ADE singularity. For example the elliptic curve $\mathbb{E}$ can be specified in Weierstrass form as $y^2 = x^3 + fx + g$, with $f, g \in K$, and upon choosing an embedding in $\mathbb{C}$, we can also associate it with a fixed choice of complex structure parameter τ. Fibering this geometry over our "4D spacetime" the arithmetic surface S, we can thus assign our gauge theory the standard parameter τ which transforms under $SL(2, \mathbb{Z})$ in the usual way:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \text{ with } ad - bc = 1 \;\; a, b, c, d \in \mathbb{Z}. \tag{8.36}$$

So, at least in principle, the effects of S-duality can be studied in this framework.

8.2.1 Arithmetic Line Operators

Having come this far, we can also ask whether we can set up the usual notions of electric and magnetic line operators in our gauge theory system. To begin, we exploit the structure of S as a fibration over $\mathrm{Spec}\, \mathcal{O}_K$ to define a formal one-form:

$$A = A_{\mathfrak{p}} d\mathfrak{p} + A_{x_{\mathfrak{p}}} dx_{\mathfrak{p}}, \tag{8.37}$$

where each $\mathfrak{p}$ refers to a maximal prime ideal of $\mathcal{O}_K$ as specified by the sheaf of differentials in $\Omega^1(\mathrm{Spec}\, \mathcal{O}_K)$ with local differential $d\mathfrak{p}$ and the fiber at each stalk comes with a differential $dx_{\mathfrak{p}}$ as defined on the curve $S_{\mathfrak{p}}$, which is just a curve over the finite field $\mathbb{F}_q \simeq \mathcal{O}_K/\mathfrak{p}$ with $q = p^r$ for some prime p and $r > 0$. We can view A as a one-form valued in $\mathfrak{sl}(N, K)$, but in which we also need to reduce each component in the stalk $A_{x_{\mathfrak{p}}}$ mod the given prime $\mathfrak{p}$.

Our first aim will be to define a notion of an electric line operator. First of all, for the vertical divisors, we are speaking of $A_{x_{\mathfrak{p}}} dx_{\mathfrak{p}}$, and so we can specify holonomies by appealing to the spectral cover construction for a curve over $\mathbb{F}_q$, namely we first take a curve Σ over $\mathbb{F}_q$, and on the spectral cover $\widetilde{\Sigma} \to \Sigma$ we take a line bundle. Under the suitable pushforward map, we then get a vector bundle, which in turn implicitly specifies a line operator for us.

So, the real issue here is to see if we can define a suitable notion of a line operator for a horizontal divisor, as specified by a section $P : \mathrm{Spec}\, \mathcal{O}_K \to S$. To do this, we first pick an embedding $K \hookrightarrow \mathbb{C}$, and order the primes $\mathfrak{p}$ according to their absolute values, viewed as standard coplex numbers. So, each prime can be viewed as living on a geometric cylinder $\mathbb{C}^\times$. We can then specify a partial ordering on $\mathrm{Spec}\, \mathcal{O}_K$, and for ease of exposition we will assume this is a total ordering (as occurs, for example, in the special case $\mathrm{Spec}\, \mathbb{Z}$). This means we

also have a notion of a path ordered exponential. The last ingredient we need to specify is a choice of representation R for $\mathfrak{sl}(N,K)$, and we denote the resulting matrices connection as $A^{(R)} = A^a T^a_{(R)}$, where the T^a are the generators of the lie algebra in the representation R. We view the restriction $A^{(R)}_{\mathfrak{p}}$ on each stalk as implicitly specified by a representation on $\mathfrak{sl}(N,\mathbb{F}_q) \simeq \mathfrak{sl}(N,\mathcal{O}_K/\mathfrak{p})$. We can then define an ordered product:

$$W_R = \prod_{\mathfrak{p}} \exp\left(\frac{2\pi i}{p} A^{(R)}_{\mathfrak{p}}\right), \tag{8.38}$$

i.e. it is an ordered product. Here that we have inserted pre-factors of $2\pi i/p$. The prime p is the characteristic of the finite residue field $\mathbb{F}_q \simeq \mathcal{O}_K/\mathfrak{p}$. We have included this factor because we need to make sure that W_R makes sense as an operator acting on the Hilbert space defined by each stalk $S_{\mathfrak{p}}$ of the arithmetic surface. In physical terms, the operator W_R is a path ordered exponential, and specifies an electric line operator.

What about the magnetic line operators? At least classically, we see what to do using our geometrically engineered setup. Indeed, since we have an elliptic curve over K with a choice of embedding in $\mathbb{C}$, we can also specify the A-cycles and S-dual B-cycles on the elliptic curve.

In the quantum setting the situation is more subtle because we view the electric line operators as order operators and the magnetic line operators as disorder operators, namely, we prescribe boundary conditions [89]. But we know precisely how to implement this prime ideal by prime ideal in $\mathcal{O}_K/\mathfrak{p}$. Indeed, what we do is examine the intersection of the image of $P : \operatorname{Spec} \mathcal{O}_K \to S$ with each vertical stalk. In the associated curve $S_{\mathfrak{p}}$, we are marking a boundary condition, as marked by a choice of representation in $\mathfrak{sl}(N,\mathbb{F}_q) \simeq \mathfrak{sl}(N,\mathcal{O}_K/\mathfrak{p})$.

There is a rather rich story explained in [89] for how these operator can act on a 2D space obtained by "dimensional reduction". There seem to be some parallels with the story we are setting up here which would be interesting to explore further.

We leave a more complete treatment of these possibilities for future work.

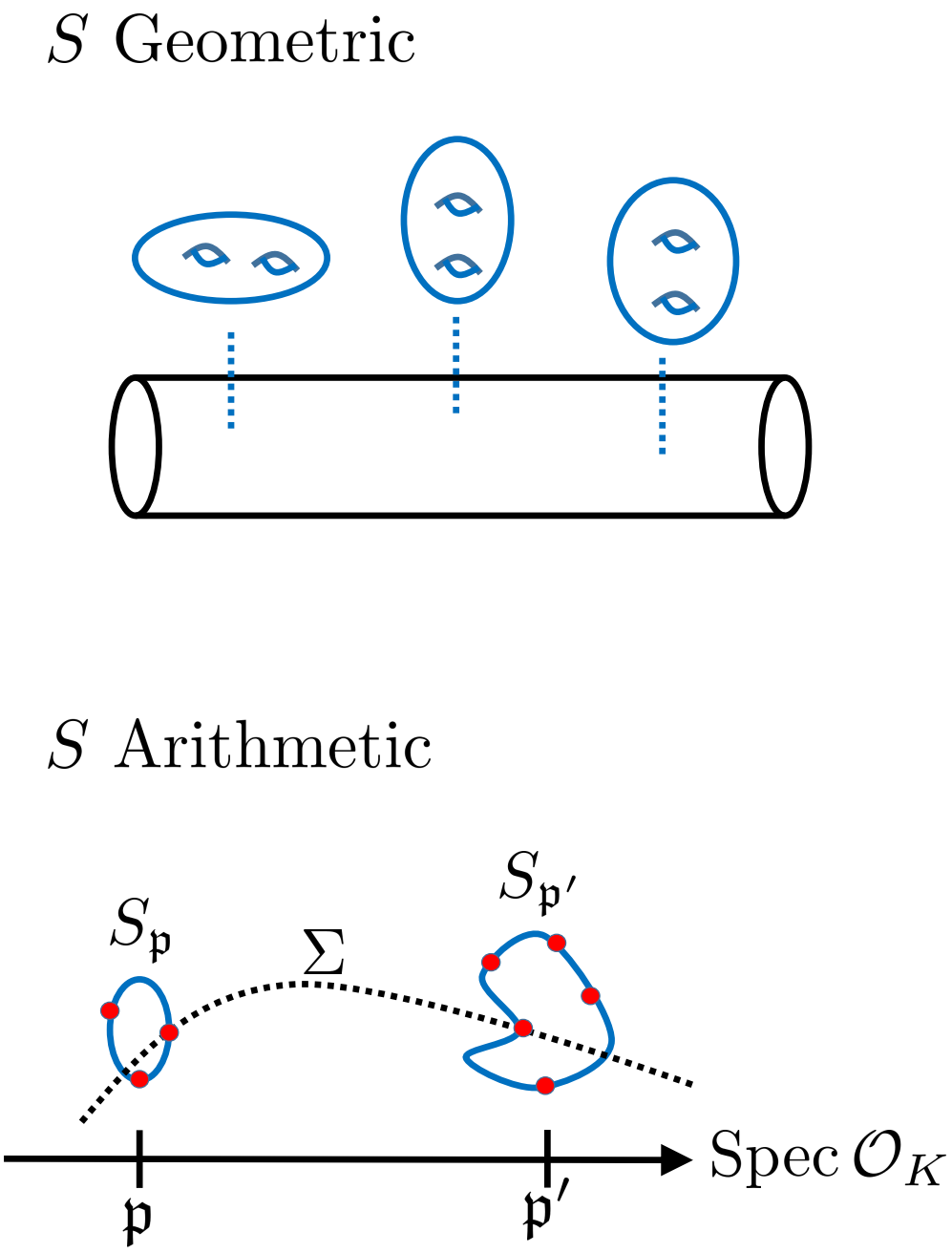

Figure 4: Depiction of a geometric surface (top) in characteristic zero, viewed as a Riemann surface fibered over a cylinder, as well as the analog for an arithmetic surface (bottom) viewed as a fibration over Spec $\mathcal{O}_K$, with K an algebraic number field and $\mathcal{O}_K$ its ring of integers. For each prime $\mathfrak{p}$, the fiber is given by the reduction of a variety over K over that prime. We have also drawn a depiction of a horizontal divisor Σ, as well as its intersection with some examples of vertical divisors such as $S_{\mathfrak{p}}$ and $S_{\mathfrak{p}'}$, with $\mathfrak{p}$ and $\mathfrak{p}'$ primes of Spec $\mathcal{O}_K$.

9 Conclusions and Further Speculations

In this note we have studied a class of physical systems in which the degrees of freedom are discretized. At some broad level, this can be phrased as taking a different choice of "natural units" in which the reduced Planck constant is instead set to the value $\hbar = N/2\pi$ with N an integer, which we view as a highly quantum regime of a physical system. From this starting point, we have shown that when $N = p$ a prime number, that the resulting physical system can be understood in terms of arithmetic geometry in characteristic p. Additionally, we have seen that this same structure persists when N is a more general integer. We have developed the analog of bosonic and fermionic degrees of freedom, and have also sketched how more general field theories can be written in characteristic p. This allowed us to present a (speculative) physical interpretation of the Hasse-Weil Zeta function. An additional feature of our considerations is that some well-established algebro-geometric correspondences appear to have close characteristic p analogs. This in turn suggests that the highly quantum regime of a string compactification may simply involve relaxing the choice of algebraic field. In the remainder of this section, we discuss some further speculations, as well as possible areas for future investigation.

One of the elements we have hinted at but have not fully developed is the structure of cohomological theories as specified by supersymmetric field theories in characteristic p. It would seem worthwhile to develop this further.

We have also presented a general expectation that some of the characteristic zero correspondences between topological field theories defined on algebraic varieties and singular local Calabi-Yau spaces extend to characteristic p. This is particularly intriguing in light of the physical formulation of the geometric Langlands program, which relies heavily on a topological twist of $\mathcal{N} = 4$ Super Yang-Mills theory [89]. From the perspective of geometric engineering, it is natural to ask whether there is a characteristic p analog of this gauge theory which could be geometrically engineered. Very speculatively, one might use this to provide a physical underpinning for some aspects of the Langlands program.

Indeed, reference [32] noted that at least for suitable flux compactifications and their relation to arithmetic Calabi-Yau threefolds, there is a notion of modularity which might persist based on the associated Zeta functions (see also [94]). In the present note we have provided some additional physical motivation for such structures. It would seem interesting to develop this further.

Perhaps more directly, we also sketched how geometric engineering can be used to provide a an operational definition of certain gauge theories on an arithmetic surface. Indeed, suitable reductions over a prime in this setting return us to the case of a Hitchin system in characteristic p, so it would seem natural to study the structure of physical notions such as S-duality and its action on electric and magnetic surface operators (perhaps along the lines of [89]) in this setting.

Continuing in the vein of possible mathematical applications, we have seen that at least

when N is a prime number p, that some physical structures can be formulated in terms of the geometry of schemes defined over the finite field $\mathbb{F}_p$ or some extension thereof. Writing $p = 2\pi\hbar$, it is natural to ask whether the physical limit $\hbar \to 1$ has any bearing on questions in arithmetic geometry. This sort of limiting procedure is sometimes mentioned in the context of what could possibly be meant by the finite field $\mathbb{F}_{\rm un}$ (see e.g. [95] for a recent discussion). It would be interesting to see whether physical considerations provide a new perspective on these questions.

The discretization of a physical theory immediately raises additional questions in the context of quantum gravity. For example, in reference [6] it was argued that Newton's constant might be quantized in units of $1/f_\pi^2$, with f_π a mass scale of a non-linear sigma model and in references [7–9], it was argued that the Fayet-Iliopoulos parameter of a supergravity theory might be quantized in units of $2M_{pl}^2$. An additional hint at the quantization of fundamental parameters appeared in [96], which argued that in appropriate decoupling limits, quantities such as $\alpha_{\rm GUT}^{-1}$ of a Grand Unified Theory (GUT) might also be discretized. The present note has taken some steps at understanding some examples of this sort, including an analysis of Planckian scale FI parameters. It would be interesting to see whether the discretization of other physical parameters can also be understood using methods from arithmetic geometry.

At a practical level, the construction we have presented has the merit of dealing with systems with finite degrees of freedom. This in turn means that numerical computations should be possible as well. It would be interesting to examine the numerical stability of such computations, especially in comparison with lattice field theories defined in Euclidean space.

We have also seen that some of our physical picture can be interpreted in terms of quantum error correcting codes. As a potential practical application of our considerations, it is natural to ask whether our path integral formalism implicitly performs a sweep over candidate quantum error correcting codes. It would be interesting to see whether this provides a way to generate new examples of quantum error correcting codes.

Our construction of field theories also highlighted that there is a characteristic p analog of the graviton, which we associate with a family of symmetric bilinear forms. We also saw that in characteristic p, there is little meaning to the "signature" of a metric since there is no ordering of elements in $\mathbb{F}_p$. Interpreting the characteristic p limit of a string compactification as the highly quantum regime of gravity, this suggests that in this discretized setting, distinctions between Lorentzian, Euclidean or more general spacetime signatures evaporate.

An intriguing feature of the present considerations is that one can view our construction at its most primitive level as specified by a Grothendieck topology for a suitable category where this a notion of "quasi-locality" even in characteristic p. From this, we saw how the structures of classical and quantum error correcting codes emerge from a suitable adaptation of a path integral defined in characteristic p. We have also sketched how standard continuum physics could emerge from these discretized considerations in a suitable large N limit. It would be interesting to develop this further.

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A 1D Lattice Systems

In this Appendix we present a brief analysis of some discretized 1D systems modulo N. We anticipate that similar formal manipulations are available in more general field theories, which would be amenable to a numerical analysis. Although we have emphasized that the proper framework for doing our computations is based on integrating over the moduli space of morphisms between schemes in characteristic p, for "practical purposes" calculating with respect to a fixed lattice field theory formulation should provide an adequate approximation for many purposes. The more general formulation seems necessary to fully capture the arithmetic geometry associated with these systems.

With this in mind, we now consider a lattice formulation for a 1D lattice system with a discretized time direction in which the reduced Planck constant satisfies:

$$\hbar = \frac{N}{2\pi}. \tag{A.1}$$

As we have already mentioned in section 2, this sort of discretization also impacts the time evolution operator, restricting us to discretized spacetimes. Our plan in this section will be to analyze a few explicit lattice systems where we take the time direction $t \in \mathbb{Z}/N\mathbb{Z}$. We take as our action:

$$S[\phi] = \sum_{t \in \mathbb{Z}/N\mathbb{Z}} L[\phi(t)] \tag{A.2}$$

with:

$$L = T - V, \tag{A.3}$$

where the kinetic term is given by a quadratic form:

$$T = \sum_{1 \leq i,j \leq N} \frac{1}{2} \Gamma_{ij} \phi(i) \phi(j). \tag{A.4}$$

and for now, we do not specify the potential $V(\phi)$. To make things concrete, we shall assume that $\underline{\Gamma}$ is an $N \times N$ matrix which takes the form of a specific 1D lattice Laplacian with

boundary conditions at the ends of the lattice:

$$\underline{\Gamma} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & ... & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \tag{A.5}$$

Where to illustrate the main ideas we have chosen $\underline{\gamma}$ so that that it does not have zero modes. We make this choice to avoid having to deal with zero modes in the kinetic term. One can of course make more general choices. Throughout our analysis we shall implicitly assume 2 does not divide N to that 2 is an element of the group $(\mathbb{Z}/N\mathbb{Z})^\times$, the multiplicative group of integers in N.

Our plan will be to analyze some aspects of this system. We mainly focus on the case of $N = p$ a prime number, but also discuss a generalization to $N = p^a$, which would correspond to a "fat point" of Spec $\mathbb{Z}$.

We first consider the case of a $D = 1$ massless free scalar, and then turn to some cases with a potential switched on. Most of the manipulations we use are covered in standard quantum mechanics and quantum field theory textbooks (see e.g [54]). We include these computations here for the reader unfamiliar with these sorts of manipulations. The main subtlety we encounter will have to do with obtaining a propagator, and analyzing the resulting correlation functions.

A.1 The $D = 1$ Free Scalar

We start with a 1D free scalar reduced modulo $N = p^a$, namely we set $V = 0$. To evaluate correlation functions, we introduce a source term $J(t)$ and study the generator function for correlation functions:

$$Z[J] = \sum_{\phi(1)\in\mathbb{Z}/N\mathbb{Z}} ... \sum_{\phi(N)\in\mathbb{Z}/N\mathbb{Z}} \exp\left(\frac{2\pi i}{N}\left(\sum_{1\leq i,j\leq N}\frac{1}{2}\gamma_{ij}\phi(i)\phi(j) + \sum_{t=1}^{N} J(t)\phi(t)\right)\right). \tag{A.6}$$

Now, in characteristic zero, it is natural to expand in "Fourier modes." This presents some complications, especially when reducing mod p^a. Rather than follow this route, we will instead stick to position space.

The first point we want to make is that the determinant of $\underline{\Gamma}$ is:

$$\det \underline{\Gamma} = N + 1 = 1 \bmod N. \tag{A.7}$$

So the inverse matrix with entries in $\mathbb{Z}/N\mathbb{Z}$ makes sense.

We now can write the action with a source term added as:

$$S[\phi] = \frac{1}{2}\Gamma_{ij}(\phi(i) + \Gamma^{-1}_{ii'}J(i'))(\phi(j) + \Gamma^{-1}_{jj'}J(j')) - \frac{1}{2}\Gamma^{-1}_{ij}J(i)J(j), \tag{A.8}$$

where in the above, we have summed over repeated indices. In the above expression we have introduced the inverse matrix M which is being computed in $\mathbb{Z}/N\mathbb{Z}$. In particular, we are viewing the entries of Γ^{-1}_{ij} as being in $\mathbb{Z}/N\mathbb{Z}$ rather than $\frac{1}{N+1}\mathbb{Z} \subset \mathbb{Q}$. This is the "natural" choice to make because all of our other quantities, including $\phi(i)$ and $J(i)$ are valued in $\mathbb{Z}/N\mathbb{Z}$.

The integrand of the generating function now takes the form:

$$\exp\left(\frac{2\pi i}{N}\left(\frac{1}{2}\Gamma_{ij}(\phi(i) + \Gamma^{-1}_{ii'}J(i'))(\phi(j) + \Gamma^{-1}_{jj'}J(j')) - \frac{1}{2}\Gamma^{-1}_{ij}J(i)J(j)\right)\right). \tag{A.9}$$

The point is that for each $\phi(i)$, we sum over all entries anyway, so the shift by $M^{-1}_{ii'}J(i')$ is "harmless". So, we can write the generating function as:

$$Z[J] = Z[0]\exp\left(\frac{2\pi i}{N}\left(-\frac{1}{2}\Gamma^{-1}_{ij}J(i)J(j)\right)\right), \tag{A.10}$$

just as we would in characteristic zero. We caution, however, that this similarity is somewhat deceptive since, for instance, the inverse of γ is computed in $\mathbb{Z}/N\mathbb{Z}$ rather than $\mathbb{Q}$.

Evaluating correlation functions now proceeds by taking functional derivatives of the sources:

$$\langle\phi(t_1)...\phi(t_m)\rangle = \left(\frac{1}{Z[0]}\frac{\hbar}{i}\frac{\delta}{\delta J(t_1)}...\frac{\hbar}{i}\frac{\delta}{\delta J(t_m)}Z[J]\right)_{J=0}. \tag{A.11}$$

As an example, we have, for $1 \leq s, t \leq N$:

$$\langle\phi(s)\phi(t)\rangle = -\frac{\hbar}{i}\Gamma^{-1}_{st} = -\frac{N}{2\pi i}\Gamma^{-1}_{st}. \tag{A.12}$$

Such expressions are, by themselves, ambiguous because operators such as $\phi(t)$ can be viewed as taking values in the integers, rather than $\mathbb{Z}/N\mathbb{Z}$. We can, however, replace these expressions by correlation functions such as:

$$\langle\exp(2\pi i\alpha\phi(s)/N)\exp(2\pi i\beta\phi(t)/N)\rangle = \exp\left(-\frac{2\pi i}{N}\alpha\beta\Gamma^{-1}_{st}\right). \tag{A.13}$$

Similar considerations hold for higher point correlation functions, via a simple application of Wick's theorem.

It is also interesting to directly analyze the behavior of the propagator $\underline{\Gamma}^{-1}$ for different choices of N. To keep things manageable, we compute the inverse for N a prime number.

To illustrate, here are the first few inverses:

$$p=3:\underline{\Gamma}^{-1}=\begin{bmatrix}0&2&1\\2&1&2\\1&2&0\end{bmatrix}=\begin{bmatrix}0&-1&1\\-1&1&-1\\1&-1&0\end{bmatrix} \tag{A.14}$$

$$p=5:\underline{\Gamma}^{-1}=\begin{bmatrix}0&4&3&2&1\\4&3&1&4&2\\3&1&4&1&3\\2&4&1&3&4\\1&2&3&4&0\end{bmatrix}=\begin{bmatrix}0&-1&-2&2&1\\-1&-2&1&-1&2\\-2&1&-1&1&-2\\2&-1&1&-2&-1\\1&2&-2&-1&0\end{bmatrix} \tag{A.15}$$

$$p=7:\underline{\Gamma}^{-1}=\begin{bmatrix}0&6&5&4&3&2&1\\6&5&3&1&6&4&2\\5&3&1&5&2&6&3\\4&1&5&2&5&1&4\\3&6&2&5&1&3&5\\2&4&6&1&3&5&6\\1&2&3&4&5&6&0\end{bmatrix}=\begin{bmatrix}0&-1&-2&-3&3&2&1\\-1&-2&3&1&-1&-3&2\\-2&3&1&-2&2&-1&3\\-3&1&-2&2&-2&1&-3\\3&-1&2&-2&1&3&-2\\2&-3&-1&1&3&-2&-1\\1&2&3&-3&-2&-1&0\end{bmatrix}. \tag{A.16}$$

As another example, here is the propagator over $N=3^2$:

$$N=3^2:\underline{\gamma}^{-1}=\begin{bmatrix}0&8&7&6&5&4&3&2&1\\8&7&5&3&1&8&6&4&2\\7&5&3&0&6&3&0&6&3\\6&3&0&6&2&7&3&8&4\\5&1&6&2&7&2&6&1&5\\4&8&3&7&2&6&0&3&6\\3&6&0&3&6&0&3&5&7\\2&4&6&8&1&3&5&7&8\\1&2&3&4&5&6&7&8&0\end{bmatrix}. \tag{A.17}$$

A.2 Adding a Mass Term

A common deformation of the free scalar involves adding a quadratic term in the physical fields. Let us consider adding a perturbation such as:

$$V(\phi)=\frac{1}{2}\lambda\phi^2, \tag{A.18}$$

which is just a mass term for the scalar. All this does is modify the diagonal entries of the kinetic term operator so that we now have:

$$\widetilde{\Gamma}_{ij}=\Gamma_{ij}-\lambda\delta_{ij}. \tag{A.19}$$

It turns out that the structure of $\widetilde{\Gamma}$ can be quite sensitive to the choice of perturbation. To see why, we again ask about the eigenvalues of our matrix $\underline{M}$, as determined by the roots of its characteristic polynomial. For example, reference [97] finds:

$$\det(\lambda \mathbb{I}_{N\times N} - \underline{\Gamma}) = U_N\left(\frac{\lambda}{2} - 1\right), \tag{A.20}$$

where $U_N(x)$ is a Chebyshev polynomial of the second kind. Restricting to $\lambda \in \mathbb{Z}/N\mathbb{Z}$, we can tabulate when this polynomial vanishes. This gives a sense of how frequently a mass parameter will end up generating a zero mode "by accident." We give the number of zeros for $N = p^a$ for p and a "small numbers":

# zeros	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$	$p=17$
p^1	1	3	3	5	7	9
p^2	1	1	1	1	1	1
p^3	1	3	3	5	7	9
p^4	1	1	1	1	1	1

(A.21)

So, we see that for $N = p$ a prime number, nearly half of the possible deformations produce a zero mode! We note that this pattern appears to also persist for $N = p^a$ for a odd.

A.3 Adding a ϕ^p Perturbation

To analyze some additionl structures in this setting, we now specialize further to the case of $N = p$ a prime number. We can consider adding a perturbation by a potential energy term. While we expect a full analysis may be difficult, there are a few simplifications which occur for specific sorts of perturbations. To illustrate, we now switch on a non-trivial potential:

$$V(\phi) = \lambda\phi^p. \tag{A.22}$$

Now, in characteristic zero, this is a challenging system to study, and a common strategy is to resort to perturbation theory in the parameter λ. In characteristic p, however, note that all elements of $\mathbb{F}_p$ satisfy:

$$\phi^p = \phi \quad \text{for} \quad \phi \in \mathbb{F}_p. \tag{A.23}$$

So, we can simplify this potential to:

$$V(\phi) = \lambda\phi. \tag{A.24}$$

The generating function for correlation functions is also straightforward to evaluate.

Adding a source term as in our previous example, the action is:

$$S = \sum_{i,j} \frac{1}{2} M_{ij} \phi\,(i)\, \phi(j) + \sum_i (-\lambda \phi(i) + \phi(i) J(i)), \tag{A.25}$$

So if we view the λ's as specifying a constant function $\lambda(i) = \lambda$ for all i, we can complete the square as before:

$$S = \frac{1}{2} M_{ij} (\phi(i) + M^{-1}_{ii'}(J(i') - \lambda(i'))(\phi(j) + M^{-1}_{jj'}(J(j') - \lambda(j')) - \frac{1}{2} M^{-1}_{ij}(J(i) - \lambda(i))(J(j) - \lambda(j)). \tag{A.26}$$

So in other words, we can just make the substitution $J \mapsto J - \lambda$.

B Lattice versus Hasse Derivatives

In this note we have alluded several times to the intuition that we can replace lattice derivatives for physical fields with derivatives of polynomials in characteristic p. In this sense, we can always view the lattice formulation as providing an approximation. We emphasize, however, that the space of polynomials retains further smooth structure which is often absent in lattice field theory.

For ease of exposition, we focus on the case of a degree m polynomial in a single variable $\phi(t) \in \mathbb{Z}[t]$, and its reduction modulo $N = p^a$ a prime power. For now, we do not restrict the degree of the polynomial $\phi(t)$ so in principle we can allow the polynomial to have degree larger than N. Recall that in the lattice formulation, we consider evaluations of the polynomial at a generic point $x \in \mathbb{Z}/N\mathbb{Z}$, and construct suitable finite differences. We can also see the appearance of derivatives of polynomials via the Taylor expansion:

$$\phi(t+x) = \sum_{r=0}^{m} \mathcal{D}^{(r)}\phi(t) \cdot x^r, \tag{B.1}$$

where $\mathcal{D}^{(r)}$ refers to the rth Hasse derivative, which acts on a monomial t^n with $0 \leq r \leq n$ as:

$$\mathcal{D}^{(r)}t^n = \frac{n!}{r!(n-r)!}t^{n-r}, \tag{B.2}$$

and zero when $r > n$. The key point for us is that for each monomial, there is at least one Hasse derivative which is non-zero, so in general, the collection of polynomials $\mathcal{D}^{(r)}\phi(t)$ for $r = 0, ..., m$ are linearly independent. Scanning over the different values of x, we get a set of linear relations between the values of $\phi(t+x)$ and the Hasse derivatives:

$$\begin{bmatrix} \phi(t) \\ \phi(t+1) \\ ... \\ \phi(t+N-2) \\ \phi(t+N-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 1 & 1^1 & ... & 1^{m-1} & 1^m \\ 1 & 2^1 & ... & 2^{m-1} & 2^m \\ 1 & (N-2)^1 & ... & ... & ... \\ 1 & (N-1)^1 & ... & (N-1)^{m-1} & (N-1)^m \end{bmatrix} \cdot \begin{bmatrix} \mathcal{D}^{(0)}\phi(t) \\ \mathcal{D}^{(1)}\phi(t) \\ ... \\ \mathcal{D}^{(m-1)}\phi(t) \\ \mathcal{D}^{(m)}\phi(t) \end{bmatrix}, \tag{B.3}$$

or more succinctly:

$$\phi(t+j) = \sum_{r=0}^{m} C_{jr}\mathcal{D}^{(r)}\phi(t), \tag{B.4}$$

where C_{jr} are the entries of an $N \times (m+1)$ matrix. The key point for us is that for degree $m = N-1$ polynomials, C is an invertible matrix, and so we can solve for the Hasse derivatives in terms of specific finite differences in the $\phi(t+j)$. Note also that if $m \geq N$, then there are in general more Hasse derivatives than finite differences. This illustrates that beyond a certain point, our lattice approximation will not work, but the formulation of physical fields as morphisms will still apply.

C Alternative Supersymmetric Action

In the main body of this note we discussed a physically motivated choice for Frobenius conjugation on fermionic fields. In this Appendix we briefly discuss the structure of the "other choice" where we instead enforce the condition:

$$F(\chi\psi) = F(\chi)F(\psi) = \chi\psi, \tag{C.1}$$

that is, we *do not* reverse the order of multiplication for products of fermion after Frobenius conjugation. This leads to some algebraic simplifications in the construction of various actions. The price we pay, however, is that there are now some new minus sign factors which must be taken into account.

We now construct an example of a 1D supersymmetric action in characteristic p, but in which our fermion products are invariant under the Frobenius automorphism. With this in mind, we now consider a single $\mathbb{F}_p$ valued bosonic field $\phi(t)$ and a pair of $\mathbb{F}_p$ valued Grassmann variables $\chi(t)$ and $\psi(t)$. We also introduce an $\mathbb{F}_p$ valued auxiliary field $f(t)$ and a superpotential $W(\phi)$ which will be a polynomial in the ϕ variable with coefficients in $\mathbb{F}_p$. We denote the derivatives of W with respect to ϕ as W' and W''. Our Lagrangian is:

$$L = \frac{1}{2}(\partial_t\phi)^2 + \chi\partial_t\psi - \frac{1}{2}f^2 + W'f + W''\chi\psi. \tag{C.2}$$

Observe that there are no factors of "i" and the products of fermions are invariant under Frobenius conjugation. An important comment is that the sign of the quadratic term of the auxiliary field has flipped sign. This does not really mean the theory has a problematic potential since one could instead write $-1 = (p-1)$. Indeed, we have already mentioned that notions such as the signature of metric lose their meaning in characteristic p.

We now verify that this Lagrangian is supersymmetric. We introduce the two variations:

$$\delta_1\phi = \psi, \quad \delta_1\psi = 0, \quad \delta_1\chi = -(\partial_t\phi + f), \quad \delta_1 f = -\partial_t\psi \tag{C.3}$$

$$\delta_2\phi = \chi, \quad \delta_2\psi = -(\partial_t\phi - f), \quad \delta_2\chi = 0, \quad \delta_2 f = +\partial_t\chi. \tag{C.4}$$

Consider first varying with respect to δ_1. This yields:

$$\begin{aligned}
\delta_1 L &= (\partial_t\phi)(\partial_t\psi) + (-\partial_t\phi - f)\partial_t\psi - (-\partial_t\psi)f && \text{(C.5)}\\
&\quad + W''(\psi)f + W'(-\partial_t\psi) && \text{(C.6)}\\
&\quad + W''(-\partial_t\phi - f)\psi && \text{(C.7)}\\
&= \partial_t\left(-W'\psi\right). && \text{(C.8)}
\end{aligned}$$

Observe that we have a "total derivative," which as we already mentioned, will be dropped (since it specifies an exact differential form).

Next, consider varying with respect to δ_2. This yields:

$$\begin{aligned}
\delta_2 L &= (\partial_t \phi)\,(\partial_t \chi) - \partial_t(-\partial_t \phi + f)\chi - (+\partial_t \chi) f && \text{(C.9)}\\
&\quad + W''(\chi) f + W'(+\partial_t \chi) && \text{(C.10)}\\
&\quad - W''(-\partial_t \phi + f)\chi && \text{(C.11)}\\
&= \partial_t((\partial_t \phi)\,\chi - f\chi + W'\chi), && \text{(C.12)}
\end{aligned}$$

which is again a "total derivative." Integrating out the auxiliary field f, we arrive at a potential for the field ϕ given by:

$$V(\phi) = -\frac{1}{2} W' W', \tag{C.13}$$

which has a sign flip relative to the characteristic zero case. This is in some sense immaterial because "positive and negative" have little meaning in the characteristic p setting. For example, we could view the "negative number" $-1 = p - 1$ as actually a "positive number."

D Evidence for Quantized FI Parameters

In this Appendix we present some evidence that quantization of FI parameters is compatible with string theory considerations. This issue has been studied using the formalism of 4D $\mathcal{N} = 1$ supergravity in references [7–9]. These considerations do not constitute a full construction, and amount to consistency conditions which would be needed in order to make sense of any putative effective field theory containing such objects. Indeed, because the size of these quantized parameters is near the Planck scale, effective field theory arguments are not fully justified. Our analysis will be similarly limited since we will be making use of notions from effective field theory, but applying them in a regime where mass scales are extremely large. Nevertheless, we find it encouraging that this analysis is compatible with such considerations.

The main idea will be to consider the $U(1)$ gauge theory associated with the worldvolume of a probe D3-brane in type IIB string theory on a spacetime of the form $M_4 \times M_6$ so that M_4 refers to the spacetime and M_6 to the six internal directions.. The configuration of branes we probe consists of a D9- anti-D9-brane pair, and a D7-brane filling M_4 and wrapping an internal four-cycle $S_{\rm GUT}$. We assume that $S_{\rm GUT}$ is contractible and is threaded by non-zero NS two-form flux and that the volume of M_6 is suitably quantized in flux units as set by the D9- anti-D9-brane pair.

In the decoupling limit of the D7-brane gauge theory, this leads to a non-commutative gauge theory on the internal directions of $S_{\rm GUT}$, as in references [98,96]. On the Higgs branch of the D3-brane probe theory, the D3-brane dissolves as an instanton, which corresponds to an anti-self-dual field strength in the internal directions of M_4. We assume that the B-flux has been chosen so that it is self-dual. In other words as explained in [99], the D3-brane sees a background of anti-D3-branes. In this case, the small instanton limit is absent, and there is instead an FI parameter ξ in the probe D3-brane, which is set by the value of the B-field.

Our first task is to estimate the value of ξ. At the origin of moduli space, the energy density for this system is:

$$E = \frac{g_{YM}^2}{2}\xi^2. \tag{D.1}$$

We can also evaluate this directly in the brane configuration: The energy density is that of a D3-anti-D3-brane annihilation. This is given by the sum of the tensions for a D3-brane and anti-D3-brane so that:

$$E = T_{D3} + T_{\overline{D3}} = 2 \times \frac{1}{g_s}\frac{1}{(2\pi)^3}\frac{1}{(\alpha')^2}. \tag{D.2}$$

The gauge coupling for a single D3-brane is also fixed by the DBI action to be $g_{YM}^2 = 2\pi g_s$. We therefore obtain our value for ξ:

$$\xi = 2 \times \frac{1}{g_s}\frac{1}{(2\pi)^2}\frac{1}{\alpha'}. \tag{D.3}$$

We now relate this value to the 4D reduced Planck mass. Recall that the 10D Newton's constant is given by:

$$16\pi G_N^{(10D)} = (2\pi)^7 g_s^2 (\alpha')^4. \tag{D.4}$$

Compactifying on a six-manifold, we obtain the 4D Newton's constant:

$$16\pi G_N^{(4D)} = \frac{(2\pi)^7 g_s^2 (\alpha')^4}{\text{Vol}(M_6)}. \tag{D.5}$$

Now, in the present setup with a D9-anti-D9-brane pair, we can compute the volume $\text{Vol}(M_6)$ in units associated with switching on a non-trivial flux in the M_4 directions which induces a Euclidean D5-anti-D5-brane pair wrapped over M_6. Hence, the natural scaling of $\text{Vol}(M_6)$ is set in units of D5-brane tension:

$$\frac{1}{\text{Vol}(M_6)} = N_{D5}\,(T_{D5} + T_{\overline{D5}}) = 2N_{D5} \times \frac{1}{g_s}\frac{1}{(2\pi)^5}\frac{1}{(\alpha')^3}, \tag{D.6}$$

where N_{D5} is a positive integer. Plugging in, we obtain the value of the 4D Newton's constant:

$$16\pi G_N^{(4D)} = 2N_{D5} \times (2\pi)^2 g_s(\alpha') = 2N_{D5} \times \frac{2}{\zeta} \tag{D.7}$$

or:

$$\zeta = N_{D5} \times \frac{4}{16\pi G_N^{(4D)}} = 2N_{D5} \times M_{pl}^2 \tag{D.8}$$

where $M_{pl}^2 = 1/8\pi G_N^{(4D)}$. We note that it is appropriate to consider a D5-brane background rather than some other (p, q) five-brane because in the duality frame being considered, D5-branes are the lowest tension five-branes available.

Let us now generalize this to FI parameters of a d-dimensional gauge theory, for $d = 2k$. In this case, a similar argument yields the relations:

$$16\pi G_N^{(d)} = g_s \times 2N_{D(9-d)} \times \frac{1}{(2\pi)^{2-d}}\frac{1}{(\alpha')^{(2-d)/2}} \tag{D.9}$$

$$\xi^2 = 4 \times T_{(d-1)}^2 \times (2\pi\alpha')^2 = 4 \times \frac{1}{g_s^2} \times \frac{1}{(2\pi)^{2d-4}} \times \frac{1}{(\alpha')^{(d-2)}} \tag{D.10}$$

which in turn implies:

$$\xi = 2 \times \frac{1}{g_s} \times \frac{1}{(2\pi)^{d-2}} \times \frac{1}{(\alpha')^{(d-2)/2}} \tag{D.11}$$

or:

$$\xi = 2N_{D(9-d)} \times \frac{1}{8\pi G_N^{(d)}}. \tag{D.12}$$

So, we see that the quantization of ξ is again in even steps of $1/8\pi G_N^{(d)}$.

E Finite Fields

In this Appendix we briefly review some aspects of finite fields. We cannot hope to provide a full review of this material, and so instead refer the interested reader to an abstract algebra textbook for further details, for example [100].

To begin, we recall that in abstract algebra, a field k has both a commutative addition and multiplication operations such that its elements form a group under addition, and after deleting 0, the identity of the additive group law, the remaining elements $k^\times$ form a multiplicative group with identity 1. Common examples include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. More abstractly, one can consider fields such as $\mathbb{Q}(t)$, $\mathbb{R}(t)$, $\mathbb{C}(t)$, with elements given by ratios of polynomials in a formal variable t. All of these examples have an infinite number of elements and specify characteristic zero fields.

One can also construct finite fields by observing that $\mathbb{Z}/p\mathbb{Z}$, the integers modulo p a prime number also satisfies all the requirements to be an algebraic field. This field is denoted as $\mathbb{F}_p$. We will shortly introduce additional finite fields $\mathbb{F}_q$ with $q = p^n$. The most important feature of all these fields is that they have characteristic p, meaning $p = 0$ in the field. Another important consequence for any characteristic p field is that we have the "Freshman's dream" equation:

$$(x + y)^p = x^p + y^p \tag{E.1}$$

This follows from expanding out the polynomial and observing that all but two coefficients are equal to zero modulo p. We also have "Fermat's little theorem" which tells us that for $m \in \mathbb{Z}$:

$$m^p \equiv m \bmod p, \tag{E.2}$$

The Frobenius map is defined by taking elements of a ring R (and thus also a field) and multiplying p times:

$$F : R \to R \tag{E.3}$$

$$r \mapsto r^p. \tag{E.4}$$

Note that for the field $\mathbb{F}_p$, all elements are fixed under this map. Consequently, we can speak of the Frobenius field endomorphism (namely one which respects addition and multiplication of the field):

$$F : \mathbb{F}_p \to \mathbb{F}_p. \tag{E.5}$$

We now introduce the finite fields $\mathbb{F}_q$. In the spirit of Galois theory, we look for the roots of irreducible polynomials over a field K. Adjoining these solutions to our original field, we obtain a field extension L, which we can view as a vector space with coefficients in K. For example, $\mathbb{C} = \mathbb{R}(i)$ where $i^2 + 1 = 0$.

Given an irreducible degree n polynomial $P_n(t)$ in the ring $\mathbb{F}_p[t]$, solving the equation:

$$P_n(t) = 0 \tag{E.6}$$

will produce a field extension of $\mathbb{F}_p$ when $n > 1$. Indeed, the condition that $P_n(t)$ is irreducible means that we can build a bigger field by adjoining the roots of $P_n(t)$ to $\mathbb{F}_p$. Denoting one such root by α so that $P_n(\alpha) = 0$, observe that $F(\alpha) = \alpha^p$ is a distinct root, since:

$$P_n(\alpha^p) = (P_n(\alpha))^p = 0, \tag{E.7}$$

where we used the Freshman's dream. Indeed, it turns out the Frobenius map generates the Galois group $\mathrm{Gal}(\mathbb{F}_p(\alpha)/\mathbb{F}_p) \simeq \mathbb{Z}/n\mathbb{Z}$, the cyclic group with n elements. Viewed as a vector space, we can treat elements of this new field as n-component vectors. But since each component has p possible entries, the total number of possible entries is p^n. This is the number of elements in the finite field $\mathbb{F}_q$ with $q = p^n$. We can again ask how the Frobenius endomorphism acts on this field. In this case, it turns out that only elements of $\mathbb{F}_p$ remain invariant. It is convenient to work in terms of a basis spanned by the images of our root α under the Frobenius map, so we can write a general element $y \in \mathbb{F}_q$ as:

$$y = y_0\alpha + y_1\alpha^p + ... + y_{n-1}F^{n-1}(\alpha). \tag{E.8}$$

Observe that any element of this bigger field satisfies the equation:

$$y^q = y, \tag{E.9}$$

which follows from the fact that F has order n on $\mathbb{F}_q$.

Continuing in this way, we can construct field extensions of $\mathbb{F}_q$ as well. We denote by $\overline{\mathbb{F}_p}$ the algebraic closure of $\mathbb{F}_p$. We note that this field also has characteristic p, but it clearly has an infinite number of elements. In this case, the Frobenius automorphism also has infinite order. One can also construct infinite order fields in characteristic p such as $\mathbb{F}_p(t)$, or more generally, the field of functions for a variety.

One feature we have used in our analysis is that we can work with V a dimension n vector space over $\mathbb{F}_p$ and interpret the action of a "dot product" in terms of an algebraic operation on $\mathbb{F}_q$. To see how this works in detail, fix a choice of a non-degenerate symmetric bilinear form:[14]

$$B : V \times V \to \mathbb{F}_p \tag{E.10}$$

$$(v, w) \mapsto B^{ij}v_iw_j. \tag{E.11}$$

Note that for any element $y \in \mathbb{F}_q$, we have an expansion in terms of powers of α as well as

[14]Here we are not using the physicist convention for upper and lower indices. We do this to avoid confusion with raising a given element to some power.

its Frobenius conjugates:

$$y = y_0\alpha + y_1\alpha^p + ... + y_{n-1}F^{n-1}(\alpha) \tag{E.12}$$
$$F(y) = y_{n-1}\alpha + y_0\alpha^p + ... + y_{n-2}F^{n-1}(\alpha) \tag{E.13}$$
$$... \tag{E.14}$$
$$F^{n-1}(y) = y_1\alpha + y_2\alpha^p + ... + y_0F^{n-1}(\alpha), \tag{E.15}$$

so by linear algebra on $\mathbb{F}_q$, we can invert this relation to write:

$$y_i = M_{ij}F^j(y), \tag{E.16}$$

for a matrix M_{ij} determined only by α. Now we can rewrite our "dot product" as a pairing:

$$B : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p \tag{E.17}$$
$$(v, w) \mapsto B^{ij}M_{ii'}F^{i'}(v)M_{jj'}F^{j'}(w), \tag{E.18}$$

where we have abused notation in treating v and w as elements of $\mathbb{F}_q$.

F Inverse Limits

In this Appendix we briefly review the notion of an "inverse limit". This material can be found in standard abstract algebra textbooks, including for example [100]. As we are mathematical dilettantes, we will content ourselves to closely follow reference [101].

We begin with a collection of groups A_i indexed by $i \in I$ such that there is a directed ordering $\geq$ on the set I. For our purposes, we can typically take this to just be the natural numbers $\mathbb{N}$. For $i \leq j$, introducing "bonding maps" given by group homomorphisms:

$$f_{ij} : A_j \to A_i \tag{F.1}$$

where we demand that f_{ii} is just the identity and $f_{ij} \circ f_{jk} = f_{ik}$ with $i \leq j \leq k$. This collection of data defines an inverse system.

To construct an inverse limit, we consider the direct product over all the A_i as well as sequences:

$$\overrightarrow{a} \in \prod_{i \in I} A_i. \tag{F.2}$$

We denote by a_i the component of the vector in A_i. The inverse limit for this system is then specified by the condition that these components of the vector are compatible with the bonding maps, i.e. we have $a_i = f_{ij}(a_j)$. The resulting set of sequences are the elements of the inverse limit:

$$\varprojlim_{i \in I} A_i \equiv \left\{ \overrightarrow{a} \in \prod_{i \in I} A_i \ \text{ such that } \ a_i = f_{ij}(a_j) \right\}. \tag{F.3}$$

The same sort of construction holds for more general sorts of maps, including ring and field homomorphisms. Since it is often clear from the context, we often leave the indexing set implicit, as we have done in the main body of the text.

G Witt Vectors

In this Appendix we briefly discuss some aspects of Witt vectors [102]. We saw the appearance of these in our brief discussion of crystalline cohomology in subsection 5.2, and again in section 8 when we discussed reduction of an integer valued action mod N. Again, as we are mathematical dilettantes, we will content ourselves to closely follow reference [103].

Given a prime number p and a commutative ring R, we denote a Witt vector as $(X_0, ..., X_m, ...)$ with $X_i \in R$. Next, introduce the Witt polynomials:

$$V^{(n)} = \sum_{i=0}^{n} p^i F^i(X_i), \tag{G.1}$$

where $F(X) = X^p$ is the Frobenius morphism. We define a ring of Witt vectors on the $V^(n)$'s, which are also known as the "ghost components." There is essentially a unique way to make the space of Witt vectors into a commutative ring such that addition and multiplication occur componentwise. In terms of two witt vectors U and V, we have:

$$(U+V)^{(i)} = U^{(i)} + V^{(i)} \tag{G.2}$$

$$(UV)^{(i)} = U^{(i)} V^{(i)}. \tag{G.3}$$

Given two Witt vectors $(X_0, X_1, ...)$ and $(Y_0, Y_1, ...)$ The explicit formulas for the first few entries of addition and multiplication are:

$$(X_0, X_1, ...) + (Y_0, Y_1, ...) = (X_0 + Y_0, X_1 + Y_1 + (X_0^p + Y_0^p - (X_0 + Y_0)^p/p), ...) \tag{G.4}$$

$$(X_0, X_1, ...) \times (Y_0, Y_1, ...) = (X_0 Y_0, X_0 Y_1^p + X_1^p Y_1 + p X_1 Y_1, ...) \tag{G.5}$$

where the appearance of "division by p" in the addition rule is just a formal way of condensing the notation for expanding out the binomial sum (no inverse powers of p appear in the final expressions).

For the purposes of this note, the main case of interest is the special case where R actually refers to a finite field such as $\mathbb{F}_p$ or $\mathbb{F}_q$. In the case of $\mathbb{F}_p$, the ring of Witt vectors is just the p-adic integers $\mathbb{Z}_p$ written in terms of Teichmüller representatives, and in the case of $\mathbb{F}_q$ it is the unramified extension of degree n of $\mathbb{Z}_p$.

Let us explain how this works in more detail for the special case of $\mathbb{Z}_p$. Recall that this space is just the elements of the p-adic numbers with p-adic norm less than or equal to one.[15] Each such element $\phi \in \mathbb{Z}_p$ can be written as a power series:

$$x = \sum_i x_i p^i, \tag{G.6}$$

[15] For a prime p, the p-adic norm of an integer $N = p^n q$ with q relatively prime to p is given by $|N|_p = 1/p^n$. This can be extended to the rational numbers. Including all limit points, we get $\mathbb{Q}_p$, the p-adic numbers.

with $a_i \in \{0, ..., p-1\}$. Now, an undesirable feature of this expansion is that the coefficients a_i do not respect the addition and multiplication rules of the Witt vectors. To get a suitable presentation, we instead use Teichmüller representatives. These are given by 0 as well as the $p-1$ roots of unity in $\mathbb{Z}_p$. Algorithmically, we begin with a p-adic integer x as in equation (G.6) and builds a new representative:

$$x = \sum_i \omega(\overline{x}_i) p^i, \tag{G.7}$$

which converges in the p-adic metric to the original sum. The algorithm for building these representatives is also straightforward, and follows from Hensel lifting / Newton's algorithm. Rather than present this in full detail, we just illustrate with an example of the algorithm in practice. The first term in the sequence is:

$$\omega(\overline{x}_0) = x_0. \tag{G.8}$$

After this, we construct $\omega(\overline{x}_1)$ by finding the unique solution of $x^{p-1} - 1 = 0 \bmod p^2$ such that $x = x_0 \bmod p$. Call this solution $\omega(\overline{x}_1)$. Next, we compute $x^{p-1} - 1 = 0 \bmod p^3$ such that $x = \omega(\overline{x}_1) \bmod p^2$. Observe that these representatives do not necessarily belong to the set $\{0, 1, ..., p-1\}$. They do, however, have the important property that $\omega(\overline{x}_i)^p = \omega(\overline{x}_i) \bmod p^{i+1}$, which is what makes the more suited to an analysis which respects Frobenius conjugation. As an example, the first few entries of the Witt vector for 2 with respect to the prime $p = 5$ are $(2, 7, 57, ...)$.

H Geometry in Characteristic p

In this Appendix we briefly discuss some aspects of geometry in characteristic p. We refer the interested reader to standard texts in algebraic geometry such as [104] for further details. Our discussion will also follow the lectures [105].

To set the stage, we recall that in algebraic geometry, we first specify a commutative ring R, and then build up an affine patch of the geometry from Spec R, the set of all prime ideals in R.[16] To illustrate, the affine complex line $\mathbb{A}^1$ can be thought of as Spec $\mathbb{C}[x]$. Indeed, the prime ideals of $\mathbb{C}[x]$ are generated by polynomials of the form $(x - c)$ for $c \in \mathbb{C}$. Each of these values of c specifies a point on our affine line. As a somewhat less intuitive example, one can even consider Spec $\mathbb{Z}$ which consists of the ideals generated by the prime integers, as well as the element 0.

There is also a notion of localizing at at a given element of Spec R. Given a prime ideal $\mathfrak{p}$, we define $R_{\mathfrak{p}}$ by first constructing the complement $\mathfrak{p}^c = R \backslash \mathfrak{p}$. Then, we are free to take inverses of $\mathfrak{p}^c$ inside R, building a new ring:

$$R_{\mathfrak{p}} = (\mathfrak{p}^c)^{-1} R. \tag{H.1}$$

One can think of this as allowing us to build fractions from objects inside R.

We are now ready to construct the cotangent space. Our discussion follows the notes of reference [105]. Given a prime ideal $\mathfrak{p} \subset R$, we get a point $[\mathfrak{p}] \in \operatorname{Spec} R$. We can then construct $R_{\mathfrak{p}}$, the localization of the ring at this point. This new ring has a maximal prime ideal $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{m}$. Observe that $[\mathfrak{p}R_{\mathfrak{p}}]$ is a point of Spec $R_{\mathfrak{p}}$. From this data, we can construct the residue field:

$$k = R_{\mathfrak{p}}/\mathfrak{m}, \tag{H.2}$$

as well as a vector space $V = \mathfrak{m}/\mathfrak{m}^2$ over the field k. The vector space V is the Zariski cotangent space at $[\mathfrak{p}]$, and we write $T^*_x X$ to denote the cotangent space of a scheme X at a point x.

This notion of cotangent space is actually quite flexible. As a particularly counter-intuitive example, we can consider Spec $\mathbb{Z}$ and calculate the derivative of integers at different primes. For example, given $40 = 2^3 \times 5$, we see that it vanishes at both the point [2] and the point [5]. Computing the derivatives at these two points yields:

$$\frac{d}{d[2]} 40 = 3 \times 2^2 \times 5 = 0 \bmod 2 \tag{H.3}$$

$$\frac{d}{d[5]} 40 = 2^3 = 3 \bmod 5. \tag{H.4}$$

[16] Recall that an ideal $I \subset R$ is defined by the properties that as an additive group, it is a subgroup of R, and that for $r \in R$ and $m \in I$, $rm \in I$. A prime ideal P is one for which if $a, b \in R$ and $ab \in P$, then either a or b is an element of P.

Our discussion so far has focussed on notions from "classical" algebraic geometry. This is enough for us to start equipping our space with appropriate sheaves, and a notion of local structure. We also mentioned in passing that even in characteristic p, there is a notion of a local analytic isomorphism as associated with a diffeomorphism. These are known as étale morphisms [106].

There are several equivalent ways to phrase this condition more precisely. We refer to a morphism of schemes $f : X \to Y$ as étale if it satisfies the condition that f is flat, locally of finite presentation, and for every $y \in Y$, the fiber $f^{-1}(y)$ is the disjoint union of points, each of which is the spectrum of a finite separable field extension of the residue field $\kappa(y)$. We found the entry [107] helpful in providing additional definitions.

Giving a full treatment would carry us to far afield from our main developments, but there is one important "moral point" to emphasize. Perhaps the most important observation is that in characteristic zero, the notion of an étale map matches up well with the condition of a map being analytic. Closely following the discussion in [108], for a map $\phi : X \to Y$ of locally finite type $\mathbb{C}$-schemes, the associated map of complex-analytic spaces $\phi^{\text{an}} : X^{\text{an}} \to Y^{\text{an}}$ is a local isomorphism if and only if ϕ is étale. In characteristic p, the main issue is in specifying the analog of the inverse function theorem.

More generally, we can speak of smooth morphisms of schemes $f : X \to Y$. A morphism f is smooth provided it is flat, finitely presented, and specified by the condition that for all $y \in Y$, $f^{-1}(y)$ is a smooth scheme over the residue fields $\kappa(y)$. One can also view smooth morphisms $f : X \to Y$ as defined by the condition that locally, they factor through an étale map $X \xrightarrow{g} \mathbb{A}^n_S \to Y$. In characteristic zero differential geometry, the smooth morphisms specify smooth submersions.

Curves in Characteristic p

We now briefly discuss curves in characteristic p. As we already mentioned, one choice is to just take $\mathbb{F}_q = \mathbb{A}^1$, the affine line. This clearly has q distinct points. One can also consider the affine line given by the zero set of the equation:

$$x + y = 0 \tag{H.5}$$

for $x, y \in \mathbb{F}_q$. Again, this is an affine line and one can verify that this also has precisely p points. More generally, we can consider cutting out a one-dimensional subspace from a hypersurface equation such as:

$$f(x, y) = \sum_{i,j} f_{ij} x^i y^j = 0, \tag{H.6}$$

for $f(x, y) \in \mathbb{F}_q[x, y]$ a polynomial in two variables. Even more generally, we can add more coordinates and consider additional intersections of hypersurfaces. Observe that as we do

this, the number of points in the ambient geometry, namely $\mathbb{F}_q^m$ also grows, becoming q^m in order. This illustrates that even in characteristic p, discretization need not mean that we are stuck with just the affine line.

It is often easier to work with hypersurfaces in a projective space. For example, for a curve in a projective space $\mathbb{P}^2$, we can write it as the zero set of the equation:

$$\left\{ f(x,y,z) = \sum_{i,j,k} f_{ijk} x^i y^j z^k = 0 \right\} \subset \mathbb{P}^2, \tag{H.7}$$

where $f(x,y,z)$ is a homogeneous polynomial in three variables.

The notion of a genus can be specified in much the same way as in characteristic zero. Indeed, for a projective curve Σ we can introduce the canonical sheaf K_Σ and then use the Riemann-Roch theorem to calculate the genus:

$$h^0(K_\Sigma) - h^1(K_\Sigma) = 2g - 2. \tag{H.8}$$

As a simple example, note that a plane curve of degree d has genus $g = (d-1)(d-2)/2$. In most well-behaved situations with a polynomial with integer coefficients reduced modulo p, this genus behaves just like its characteristic zero counterpart, though there are some notable exceptions. As a pathological example, note that the equation $y = x^q + x = 2x$ for $x, y \in \mathbb{F}_q$.

One can also ask about the number of points in this curve. For a curve Σ defined over $\mathbb{F}_q$, there is also an important Hasse-Weil bound on the number of solutions (see e.g. [16]):

$$|\#\Sigma(\mathbb{F}_q) - (q+1)| \leq 2g\sqrt{q}. \tag{H.9}$$

I Some Zeta Functions

In this Appendix we collect a few examples of Zeta functions. To begin, we fix our base field to be $\mathbb{F}_q$, and assume (as usual) that q is odd for simplicity. As a first example, consider the variety $\mathbb{A}^n$, i.e. affine n-dimensional space. Counting points in this setting is straightforward, and we get:

$$Z_{\mathbb{A}^n,q}(z) = \frac{1}{1-q^n z}. \tag{I.1}$$

The case of projective n-dimensional space is similar, and gives:

$$Z_{\mathbb{P}^n,q}(z) = \frac{1}{(1-z)(1-qz)...(1-q^n z)}. \tag{I.2}$$

As a somewhat more involved example discussed in reference [63], we next consider the elliptic curve $\mathbb{E}$ defined as a zero set in $\mathbb{P}^2$:

$$Y^2 Z = X^3 + XZ^2, \tag{I.3}$$

in the obvious notation. The Zeta function in this case is:[17]

$$Z_{\mathbb{E},q}(z) = \frac{1-at+qt^2}{(1-t)(1-qt)}, \tag{I.4}$$

where the number a is implicitly fixed by the relation:

$$\#(\mathbb{E}) = -a + 1 + q, \tag{I.5}$$

where $\#(\mathbb{E})$ is the number of points in $\mathbb{E}$ defined over $\mathbb{F}_q$. Note also that the denominator is the same as that of $\zeta(\mathbb{P}^n, z)$. An additional remark here is that for this curve, the rigid cohomology group is [63, 64]:

$$H^1_{\text{rig}}(\mathbb{E}) \simeq \mathbb{Q}_q \frac{dx}{y} \oplus \mathbb{Q}_q x \frac{dx}{y}, \tag{I.6}$$

where $\mathbb{Q}_q$ with $q = r$ denotes the degree r unramified extension over the p-adics $\mathbb{Q}_p$, i.e. we have $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \simeq \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$.

One can also consider the affine case, i.e. by setting $Z = 1$ in equation (I.3) which yields [63]:

$$Z_{\mathbb{E}_{\text{aff}},q}(z) = Z(\mathbb{E}, z)(1-z)^2(1-z^2), \qquad \text{if } q \equiv -1 \bmod 4 \tag{I.7}$$

$$Z_{\mathbb{E}_{\text{aff}},q}(z) = Z(\mathbb{E}, z)(1-z)^4, \qquad \text{if } q \equiv +1 \bmod 4. \tag{I.8}$$

[17] There is an unfortunate clash of notation between the projective coordinate and the Zeta function. It should be clear from the context which is meant.

J Topologies

In this Appendix we briefly review some aspects of Grothendieck topologies. We refer the interested reader to [109] for further elaboration on the subject. For our purposes, the main point of these notions is to provide a suitable generalization of covering spaces which can produce a non-trivial cohomology theory, even in the discretized situation present in defining varieties over finite fields. Essentially quoting from [109], one defines a topology or site T as a category $\mathrm{cat}(T)$ of a set of coverings $\mathrm{cov}(T)$ defined as families of morphisms $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ in $\mathrm{cat}(T)$ such that the three properties hold:

- (T1) For $\{U_i \to U\}$ in $\mathrm{cov}(T)$ and a morphism $V \to U$ in $\mathrm{cat}(T)$, all fiber products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}$ is again in $\mathrm{cov}(T)$.
- (T2) Given $\{U_i \to U\} \in \mathrm{cov}(T)$, and a family $\{V_{ij} \to U_i\} \in \mathrm{cov}(T)$ for all $i \in I$, the family $\{V_{ij} \to U\}$ obtained by composition of morphisms also belongs to $\mathrm{cov}(T)$.
- (T3) If $\varphi : U' \to U$ is an isomorphism in $\mathrm{cat}(T)$ then $\{U' \xrightarrow{\varphi} U\} \in \mathrm{cov}(T)$.

Again, the point of these notions is to have a sense of open coverings as one has in standard topology, but in which the emphasis is on the morphisms rather than the sets themselves. Standard notions of presheaves and sheaves can be defined in this setting as well. For example, letting $\mathcal{C}$ denote a category of products (which can include the case of the category of abelian groups or the category of sets), and T a topology, we can define a presheaf on T with values in $\mathcal{C}$ as a contravariant functor $F : T \to \mathcal{C}$. We can then speak of a sheaf on T as defined by the condition that if for every covering $\{U_i \to U\}$ in T, the following diagram is exact:

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j), \tag{J.1}$$

namely the suitable restrictions defined by the equalizer map $\rightrightarrows$ are compatible with one another. In the above, $U_i \times_U U_j$ is the standard fiber product as specified by the commutative diagram:

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U \end{array} \tag{J.2}$$

where the structure of the exact sequence of line (J.1) now follows since F is a contravariant functor.

The main usage of this formalism for us is in defining the various étale topologies. In particular, given a scheme X, we can specify the category of étale X schemes, denoted by

Et/X. The (small) étale site (i.e. topology) of X is denote by $X_{\text{ét}}$, where $\text{cat}(X_{\text{ét}})$ is just Et/X and the space of coverings $\text{cov}(X_{\text{ét}})$ is the set of surjective families of morphisms in Et/X. One can also speak of a big étale site, but at the level of cohomology, these distinctions are often not important, and we will not elaborate on this further.

The notion of a crystalline site also implicitly appears in our discussion. From [110], if X is a scheme over a field k, then the crystalline site of X relative to W_n, denoted $Cris(X/W_n)$, has as its objects pairs $U \to T$ consisting of a closed immersion of a Zariski open subset U of X into some W_n-scheme T defined by a sheaf of ideals J, together with a divided power structure on J compatible with the one on W_n.

Given a suitable notion of global sections for sheaves construct the associated cohomology theory via right-derived functors of the global sections. Given a sheaf F and open cover $\{U_i \to U\}_{i\in I}$, we consider the left-exact functor $F \mapsto H^0(\{U_i \to U\}_{i\in I}, F)$. Then, the right-derived functor provides a definition of the higher degree cohomology groups: $H^j(U, \mathcal{F}) = R^j(U, \mathcal{F})$ (see e.g. [109]). Using the sheaf property, we can then extend to X.

For our purposes, the utility of introducing the étale topology is that we also have the Artin comparison theorem [111], which states, for an algebra A given for example by either a finite field $\mathbb{F}_q$, $\mathbb{Z}_p$, and $\mathbb{Q}_p$ (and suitable generalizations thereof) that:

$$H^\bullet(X_{\text{ét}}, A) \simeq H^\bullet(\mathcal{X}^{\text{an}}, A), \tag{J.3}$$

where $\mathcal{X}^{\text{an}}$ refers to the analytification of X over $\mathbb{C}$, i.e. we interpret our variety as defined over $\mathbb{C}$ and then equip it with the standard topology of an analytic space.

K Codes

In this Appendix we review some aspects of classical and quantum codes alluded to earlier. For a review of linear and non-linear codes, see for example the thesis [112]. For a review of algebraic geometry codes, see for example reference [113]. For a review of how quantum codes can be obtained from algebraic codes, see reference [114]. Our plan will be to first review some aspects of classical coding theory, and in particular the relation to algebraic curves over finite fields. We then turn to quantum codes obtained from these classical codes. As throughout, we let q denote a power of some prime p.

K.1 Classical Algebraic Codes

To set the stage, let us recall that the main idea in much of classical coding theory is to send messages over a noisy channel. Our discussion will follow that given in reference [112]. More precisely, one has in mind the following schematic diagram:

$$\underbrace{[\text{Source}] \to [\text{Transmitter}]}_{\text{Input}} \to \underbrace{[\text{Receiver}] \to [\text{Sink}]}_{\text{Output}}. \tag{K.1}$$

The "source" and "sink" may consist of k different possible messages which are then encoded in a larger set of n "codewords" for the transmitter and receiver. The main idea is that by a suitable embedding of these k possible messages in the codewords, random errors in the transmission can be minimized. Of course, one way to proceed is to encode all information in a string of 1's and 0's, but more generally, our basic alphabet may consist of a q-ary code with q different possible letters, as for example would occur if we use the finite field $\mathbb{F}_q$. A code is then some collection of different codewords of length n. The main idea is that in passing a message from the transmitter to the receiver, there may be some noise, i.e. errors may be generated. These errors are just vectors $e \in \mathbb{F}_q^n$ so that for an element $c \in C$, the error is just given by $c + e$, i.e. it flips some of the entries in our codeword. Much of the art of the subject revolves around finding efficient ways to protect messages so that even when errors are present, the message can be decoded. Along these lines, We refer to the redundancy of a code as $n - k$ and the information rate of a code as:

$$R = \frac{1}{n} \log_q |C| \,. \tag{K.2}$$

We can speak of the weight of a codeword as the number of entries which are different from zero. Given a code C, we can also specify the distance d by computing the Hamming distance of elements in the image set $C(V)$ as specified by taking an input word in V and encoding it in C:

$$d(C) = \min_{v \neq v'} \{\text{dist}_{\text{Ham}} C(v), C(v') \text{ with } v, v' \in V\} \,, \tag{K.3}$$

where dist_{Ham} is the "Hamming distance", i.e. we view our words as elements of $\mathbb{F}_q^n$ with respect to a fixed basis and count the number of entries over which the two vectors are different. When the context is clear, we shall often just write C to denote the code space.

Specializing further, we can build a large set of codes by assuming the various words and codes are built from vectors in vector spaces over $\mathbb{F}_q$. In this case, the source is just a k-dimensional vector space $V \simeq \mathbb{F}_q^k$ and the code space is an n-dimensional vector space $W \simeq \mathbb{F}_q^n$. We can speak of the embedding $V \to W$ as specifying a code C, i.e. it is just the image set of V inside W. Given a linear code C, we can also specify the distance d by computing the Hamming distance of elements in the image set $C(V)$:

$$d(C) = \min_{v \neq v'} \{\text{dist}_{\text{Ham}} C(v), C(v') \text{ with } v, v' \in V\}, \tag{K.4}$$

where dist_{Ham} is the "Hamming distance", i.e. we pick a basis for $\mathbb{F}_q^n$ and count the number of entries over which the two vectors are different. Note that $d(C)$ does not depend on this choice of basis since we always minimize over all vectors in the image anyway. We refer to a linear $[n, k, d]_q$ code where $d(C) = d$ is the minimum distance of the code. In this case, the information rate is just $R = k/n$.

One can also introduce a notion of non-linear codes. Treating each codeword as an element of $\mathbb{F}_q^n$ with respect to a fixed basis, we can view this as defining a set $C_{\text{nl}} \subset \mathbb{F}_q^n$, where the subscript serves to remind us that this is not a vector space. We refer to the kernel of this space as $K(C_{\text{nl}})$:

$$K(C_{\text{nl}}) = \{v \in C_{\text{nl}} \text{ with } \lambda v + C_{\text{nl}} = C_{\text{nl}} \text{ for all } \lambda \in \mathbb{F}_q\}, \tag{K.5}$$

which is a vector space. The rest of the codewords can then be obtained by adding appropriate vectors, i.e. by suitable affine transformations:

$$C_{\text{nl}} = \bigcup_{i=1}^{t} (K(C_{\text{nl}}) + v_i), \tag{K.6}$$

that is, we introduce t coset vectors $v_1, ..., v_t$ to build the rest of the codewords.

Algebraic varieties over finite fields provide a way to build examples of codes, a topic we now review following [113]. Intriguingly, the additional geometric structure present in this class of examples often provide a way to build "good" examples in the sense that certain information theoretic quantities can be handled analytically. We cannot hope to provide a full characterization of the subject, but we can at least explain how these geometric ingredients emerge.

To keep things as concrete as possible, we fix X a smooth, projective irreducible curve of genus g over the finite field $\mathbb{F}_q$. We begin by splitting up introducing two sets of points which we write as $P_1, ..., P_n$ and $Q_1, ..., Q_m$. From these, we can form the divisors $D = P_1 + ... + P_n$ and $G = Q_1 + ... + Q_m$. We can then introduce the Riemann-Roch vector space associated

with this divisor:

$$\mathcal{L}(G) = \{f \in \mathbb{F}_q(X) \quad \text{such that} \quad (f) + G \geq 0\} \cup \{0\}\,. \tag{K.7}$$

At this point, it is helpful to recall that for a plane curve in $\mathbb{P}^2$ defined by the equation $h(x, y, z) = 0$ with x, y, z homogeneous coordinates, the space of functions is given by ratios of the form $p(x, y, z)/q(x, y, z)$ where h does not divide either p or q. This property ensures that $\mathcal{L}(G)$ is finite dimensional.

We note that $\mathcal{L}(G)$ is also just the space of global sections for a line bundle, and in this case it is customary to denote it as $H^0(X, \mathcal{O}_X(G))$. This defines a vector space which we denote by $\ell(G) = k$. We can introduce a basis which we write as $\{f_a\}$ for $a = 1, .., k$. Given these functions, we can produce a code by evaluating at all n points of D:

$$\text{ev}_D : \mathcal{L}(G) \to \mathbb{F}_q^n \tag{K.8}$$

$$f \mapsto (f(P_1, ..., P_n). \tag{K.9}$$

Doing so, we get an $n \times k$ matrix as specified by $f_a(P_i)$ where $a = 1, ..., k$ and $i = 1, ..., n$. This specifies what is known as an m-point code, and is often denoted as $C_{\mathcal{L}}(D, G)$. We note that if $\deg G < n$, then this specifies an $[n, k, d]_q$ linear code with n set by the number of evaluation points, $k = l(G)$ the dimension of the linear system, and $d = n - \deg G$ the minimal distance between codewords.

In the literature on algebraic geometry codes, it is also customary to discuss the space of meromorphic one forms:

$$\Omega(G - D) = \{\omega \in \Omega(X) \quad \text{such that} \quad (\omega) \geq G - D\} \cup \{0\}\,. \tag{K.10}$$

In this case, one specifies a code by computing the residues of ω at the marked points. In other words, the evaluation map in this case is given by:

$$\text{ev}_D : \Omega(G - D) \to \mathbb{F}_q^n \tag{K.11}$$

$$\omega \mapsto (\text{res}_{P_1}\omega, ..., \text{res}_{P_n}\omega), \tag{K.12}$$

and the corresponding linear code is denoted by $C_\Omega(D, G)$. Let us note that there is a duality between the codes $C_{\mathcal{L}}(D, G)$ and $C_\Omega(D, G)$ which is often written as:

$$C_{\mathcal{L}}(D, G) = C_\Omega(D, G)^\perp, \tag{K.13}$$

where we have implicitly a notion of orthogonality as induced by introducing a pairing on $\mathbb{F}_q^n$ given by:

$$\mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q \tag{K.14}$$

$$(a, b) \mapsto a \cdot b = \sum_{i=1}^{n} a_i b_i, \tag{K.15}$$

and for a vector space $V \subset \mathbb{F}_q^n$, we define

$$V^{\perp} = \left\{ w \in \mathbb{F}_q^n \quad \text{such that } w \cdot v = 0 \quad \text{for all } v \in V \right\}. \tag{K.16}$$

Now we can see why equation (K.13) is true; We can consider any $f \in \mathcal{L}(G)$ and any $\omega \in \Omega(G)$. The dot product between the two is, in the obvious abuse of notation:

$$(f, \omega) = \sum_{i=1}^{n} f(P_i) \operatorname{res}_{P_i} \omega = \sum_{i=1}^{n} \operatorname{res}_{P_i} f\omega = 0, \tag{K.17}$$

where the last equality follows from the fact that we are summing over *all* the residues of a compact curve.

Though we will not be too concerned with "practical applications," some important properties of algebraic codes include the fact that infinite families of codes $[n_i, k_i, d_i]_q$ can be constructed such that the information rate $R_i = k_i/n_i$ and relative distance $\delta_i = d_i/n_i$ remain finite and non-zero as $i \to \infty$.

The above considerations can also be extended to produce a class of non-linear codes which again have good asymptotic coding properties. Following [115] we next consider a stable vector bundle E over the curve X. Stability is defined in essentially the same way as in characteristic zero; we first change base to the algebraic closure $\overline{\mathbb{F}}_q$ and then specify the slope as $\mu(E) = \deg(E)/\mathrm{rk}(E)$, where $\deg(E)$ denotes the degree and $\mathrm{rk}(E) \equiv r$ the rank of the vector bundle. We refer to a bundle as being stable if and only if, for every E' a subbundle of E, we have $\mu(E') < \mu(E)$. Now, the important point for us is that for any point $x \in X$, we can consider the stalk E_x which is just a copy of $\mathbb{F}_q^r \simeq \mathbb{F}_Q$, with and $Q = q^r$. We can then proceed much as we did in the line bundle case, we simply consider the evaluation map at n different points of the global sections of E:

$$\mathrm{ev} : H^0(X, E) \to \bigoplus_{i=1}^{n} E_{P_i} \simeq \mathbb{F}_Q^n \tag{K.18}$$

$$v \mapsto (v(P_1), ..., v(P_n)). \tag{K.19}$$

Observe that this need not define an $\mathbb{F}_Q$ linear code, but we do get an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_Q^n$. Note that if the evaluation map is injective, then the size of the code is $K = q^{h^0(X,E)}$, the dimension of the non-linear code is $k = \log_Q K$, and since our code space is $\mathbb{F}_q$-linear, d is just the minimal weight of a non-zero codeword.

Specifying codes is in some sense the "easier" part of the story. Indeed, the utility of a given code also requires one to be able to efficiently decode a given signal, and there is again a vast literature centered around how to do this. We will not dwell on this point since it is

beyond the scope of the present considerations.

K.2 Quantum Error Correcting Codes

Let us now turn to the case of quantum error corrrecting codes generated from algebraic codes. Our discussion follows reference [114]. To begin, we recall that a q-ary quantum code of length n is specified as a dimension k complex subspace $Q \subset (\mathbb{C}^q)^{\otimes n}$. We can adopt a basis of states $|u_1, ..., u_n\rangle$, where each $u_i \in \mathbb{F}_q$ specifies a single qudit register. We can introduce a set of unitary operators E_i which serve to encode possible errors made in transmitting information. There is then an orthogonal decomposition of $(\mathbb{C}^q)^{\otimes n}$ as:

$$(\mathbb{C}^q)^{\otimes n} \simeq \bigoplus_{i=0}^{t} E_i Q, \tag{K.20}$$

where E_0 is the identity, and $t = q^{n-k} - 1$.

Given a quantum code Q with basis $|\psi_1\rangle, ..., |\psi_k\rangle$, we have a notion of errors E and F being "correctable" if they are distinguishable, namely if the following conditions are met:

$$\langle \psi_i | E^\dagger F | \psi_j \rangle = 0 \quad \text{and} \quad \langle \psi_i | E^\dagger F | \psi_i \rangle = \langle \psi_j | E^\dagger F | \psi_j \rangle . \tag{K.21}$$

for all i, j.

A basis of error operators can be specified as follows. Begin by considering the special case where $n = 1$. Then, we note that for some $m \geq 1$, we have $\mathbb{F}_q$ is a vector space over $\mathbb{F}_p$, that is, $q = p^m$. Introduce $a, b, u \in \mathbb{F}_q$. Given a state $|u\rangle \in \mathbb{C}^q$, we have the qudit operations:

$$T_a |u\rangle = |u + a\rangle \quad \text{and} \quad R_b |u\rangle = \xi^{\mathrm{Tr}(bu)} |u\rangle , \tag{K.22}$$

where ξ is a primitive pth root of unity. Here, we have again used the trace map Tr: $\mathbb{F}_q \to \mathbb{F}_p$ to ensure that all phases have unit modulus. We can then define an "error operation" $E_{ab} = T_a R_b$, and the span of these operators constitute the set of errors on a single qudit.

Next consider the case $n > 1$. By abuse of notation we let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and $u = (u_1, ..., u_n)$. Then, we can introduce error operators:

$$E_{ab} = T_a R_b \tag{K.23}$$

where:

$$T_a = \underbrace{T_{a_1} \otimes ... \otimes T_{a_n}}_{n} \quad \text{and} \quad R_b = \underbrace{R_{b_1} \otimes ... \otimes R_{b_n}}_{n}, \tag{K.24}$$

and we have the error basis:

$$\mathcal{E}_n = \left\{ E_{ab} = T_a R_b \quad \text{with} \quad a, b \in \mathbb{F}_q^n \right\}, \tag{K.25}$$

and the error group:

$$\mathcal{G}_n = \left\{ \xi^i E_{ab} \quad \text{with} \quad a, b \in \mathbb{F}_q^n \quad \text{and} \quad 0 \leq i \leq q-1 \right\}. \tag{K.26}$$

We can also speak of the weight of an error $\xi^i E_{ab}$ as the number of operations different from the identity, i.e.:

$$\text{wt}(\xi^i E_{ab}) = n - \left| \{ i : a_i = b_i = 0 \} \right|, \tag{K.27}$$

and thus, can speak of the minimum distance of a quantum code Q as the maximum weight of error operations which can be corrected

$$d(Q) = \max_{\substack{|u\rangle, |v\rangle \in C \\ E \in \mathcal{G}_n}} \{ d \text{ such that } \langle u|v\rangle = 0 \text{ and } \text{wt}(E) \leq d-1 \Rightarrow \langle u|E|v\rangle = 0 \}. \tag{K.28}$$

We can now refer to a quantum code of length n, dimension k and minimum distance d as an $[[n, k, d]]_q$ code.

K.2.1 Stabilizer Codes and Algebraic Curves

Let us now specialize a bit further. Our aim will be to introduce quantum stabilizer codes and how to construct them using algebraic curves over finite fields. For disussion of quantum stabilizer codes, see reference [116].

The stabilizer of a given a subgroup of $S \subset \mathcal{G}_n$, provides a possible quantum analog to linear codes. We define a q-ary quantum stabilizer code C of length n as the joint eigenspace of operators, i.e.:

$$Q = \text{Stab}(S) = \left\{ |u\rangle \in (\mathbb{C}^q)^{\otimes n} \quad \text{with} \quad M|u\rangle = |u\rangle \quad \text{for all } M \in S \right\}. \tag{K.29}$$

The interesting point for us is that algebraic curves over finite fields provide a natural way to specify subgroups of $S \subset \mathcal{G}_n$, and consequently, give us a way to build quantum stabilizer codes. This is often referred to as the CSS construction, after references [49, 50].

To begin, introduce a quadratic field extension $\mathbb{F}_{q^2} = \mathbb{F}_q(\omega)$ over $\mathbb{F}_q$. We can take as basis vectors ω and $\omega^q \equiv \overline{\omega}$ and present elements of $\mathbb{F}_{q^2}^n$ as linear combinations $\omega a + \overline{\omega} b$ for $a, b \in \mathbb{F}_q^n$. Given this, we can perform the following composition of maps:

$$\mathbb{F}_{q^2}^n \xrightarrow{f} \mathbb{F}_q^{2n} \xrightarrow{g} \mathcal{G}_n \tag{K.30}$$

$$\omega a + \overline{\omega} b \mapsto (a; b) \mapsto E_{ab}. \tag{K.31}$$

So, given a suitable linear code subspace $C \subset \mathbb{F}_q^{2n}$, the image space $g(C)$ gives a collection of error correction operations. The stabilizer $Q = \text{Stab}(g(C))$ then defines a quantum stabilizer code.

To make use of this in explicit constructions, we can introduce a notion of an orthogonal dual, as specified by making a choice of bilinear pairing. Two canonical options are a symplectic pairing and a Hermitian pairing. For the symplectic pairing, we use $\mathbb{F}_{q^2}^n \simeq \mathbb{F}_q^{2n}$ and assume $u = \omega a + \overline{\omega} b$ and $v = \omega a' + \overline{\omega} b'$. Then, we can specify the product as:

$$(\omega a + \overline{\omega} b) *_s (\omega a' + \overline{\omega} b') = \mathrm{Tr}\left(\sum_{i=1}^{n} a_i b_i' - b_i a_i' \right). \tag{K.32}$$

Given $u, v \in \mathbb{F}_{q^2}^n$, we denote the Hermitian pairing as the $\mathbb{F}_q$ valued expression:

$$u *_h v = \sum_{i=1}^{n} u_i v_i^q = \sum_{i=1}^{n} u_i \overline{v_i}. \tag{K.33}$$

Observe that orthogonality in the Hermitian pairing implies orthogonality with respect to the symplectic pairing.

Given a vector space $C \subset \mathbb{F}_q^{2n}$, we can then consider, for each choice of pairing, the space of vectors which are "orthogonal" to C with respect to the symplectic pairing as:

$$C^{(s)} = \left\{ v \in \mathbb{F}_q^{2n} \quad \text{such that} \quad v *_s c = 0 \text{ for all } c \in C \right\} \tag{K.34}$$

Given a vector space $C \subset \mathbb{F}_{q^2}^n$, we can reference the space of vectors which are "orthogonal" to C with respect to the Hermitian pairing as:

$$C^{(h)} = \left\{ v \in \mathbb{F}_{q^2}^n \quad \text{such that} \quad v *_h c = 0 \text{ for all } c \in C \right\}. \tag{K.35}$$

An important general result due to reference [51] is that with $q = p^m$ and for $C \subset \mathbb{F}_q^{2n}$ an $\mathbb{F}_p$-linear code of order p^r which is self-orthogonal with respect to the $*_s$ product, namely $C \subset C^{(s)}$, then any eigenspace of the CSS map $g(C)$ is a $[[n, n - \frac{r}{m}, d(C^{(s)} \backslash C)]]_q$ code. Note that we implicitly have r/m an integer, and in many applications we shall make the further assumption that C is an $\mathbb{F}_q$-linear code, defining a vector space of dimension $k = r/m$.

There are various immediate corollaries of this result. For example, can now specify two linear codes C_1 and C_2 of length n and respective dimensions k_1 and k_2 with $C_1 \subset C_2$. Then, using the standard dot product of line (K.15), we construct the dual space $C_2^{\perp}$. We can then produce a subspace $C = \omega C_1 + \overline{\omega} C_2^{\perp} \subset \mathbb{F}_{q^2}^n$. Observe that $f(C) = D$ is self-orthogonal, with $f : \mathbb{F}_{q^2}^n \to \mathbb{F}_q^{2n}$ defined in line (K.30). This can be used to prove that the eigenspace of $g(D)$ is in fact a quantum stabilizer code, and implicitly specifies an $[[n, k, d]]_q$ code with $k = k_2 - k_1$ and $d = \min\{d(C_2 \backslash C_1), d(C_1^{\perp} \backslash C_2)\}$.

We can obtain a similar set of assertions using the Hermitian pairing and a classical q^2-ary linear code $[n, k, d]_{q^2}$, where we assume that the associated vector space C is self-orthogonal with respect to the Hermitian pairing. Denoting this orthogonal space by $C^{(h)}$, the resulting quantum code constructed from this data is an $[n, n - 2k, \min\{\mathrm{wt}(C^{(h)} \backslash C)\}]]_q$ code.

Having seen how to build quantum stabilizer codes from classical codes, we next observe that classical algebraic codes provide a way to generate many examples of such quantum stabilizer codes. Additionally, some of the conditions implcitly used, such as the condition that we find two linear codes C_1 and C_2 such that $C_1 \subset C_2$ simply amount to specifying line bundles in the appropriate fashion. For example, using the fact that for divisors $A \leq B$ (namely $B - A$ is effective) we observe that the line bundles satisfy $\mathcal{L}(A) \subset \mathcal{L}(B)$, and so we also have $C_{\mathcal{L}}(D, A) \subset C_{\mathcal{L}}(D, B)$ for the corresponding linear codes, where $D = P_1 + ... + P_n$ is the divisor given by our "evaluation points".

Summarizing, we have discussed a few ways to generate quantum stabilizer codes. In fact, the interesting feature of these examples is that we also establish the existence of families of quantum codes, since we already have families of classical linear codes.

References

[1] C. Vafa, "The String Landscape and the Swampland," arXiv:hep-th/0509212.

[2] J. J. Heckman and C. Vafa, "Fine Tuning, Sequestering, and the Swampland," *Phys. Lett.* **B798** (2019) 135004, arXiv:1905.06342 [hep-th].

[3] E. Verlinde and H. Verlinde, "Black Hole Entanglement and Quantum Error Correction," *JHEP* **10** (2013) 107, arXiv:1211.6913 [hep-th].

[4] A. Almheiri, X. Dong, and D. Harlow, "Bulk Locality and Quantum Error Correction in AdS/CFT," *JHEP* **04** (2015) 163, arXiv:1411.7041 [hep-th].

[5] A. Milekhin, "Quantum Error Correction and Large N," arXiv:2008.12869 [hep-th].

[6] E. Witten and J. Bagger, "Quantization of Newton's Constant in Certain Supergravity Theories," *Phys. Lett. B* **115** (1982) 202–206.

[7] N. Seiberg, "Modifying the Sum Over Topological Sectors and Constraints on Supergravity," *JHEP* **07** (2010) 070, arXiv:1005.0002 [hep-th].

[8] J. Distler and E. Sharpe, "Quantization of Fayet-Iliopoulos Parameters in Supergravity," *Phys. Rev. D* **83** (2011) 085010, arXiv:1008.0419 [hep-th].

[9] S. Hellerman and E. Sharpe, "Sums over topological sectors and quantization of Fayet-Iliopoulos parameters," *Adv. Theor. Math. Phys.* **15** (2011) 1141–1199, arXiv:1012.5999 [hep-th].

[10] Döring, A. and Isham, C.J., "A Topos foundation for theories of physics. I. Formal languages for physics," *J. Math. Phys.* **49** (2008) 053515, arXiv:quant-ph/0703060.

[11] Döring, A. and Isham, C.J., "A Topos foundation for theories of physics. II. Daseinisation and the liberation of quantum theory," *J. Math. Phys.* **49** (2008) 053516, arXiv:quant-ph/0703062.

[12] Döring, A. and Isham, C.J., "A Topos foundation for theories of physics. III. The Representation of physical quantities with arrows," *J. Math. Phys.* **49** (2008) 053517, arXiv:quant-ph/0703064.

[13] Döring, A. and Isham, C.J., "A Topos foundation for theories of physics. IV. Categories of systems," *J. Math. Phys.* **49** (2008) 053518, arXiv:quant-ph/0703066.

[14] Döring, Andreas and Isham, Chris, "'What is a Thing?': Topos Theory in the Foundations of Physics," *Lect. Notes Phys.* **813** (2010) 753–937, arXiv:0803.0417 [quant-ph].

[15] C. J. Isham, "Topos Methods in the Foundations of Physics," arXiv:1004.3564 [quant-ph].

[16] A. Weil, "Numbers of solutions of equations in finite fields," *Bull. Amer. Math. Soc.* **55** no. 5, (05, 1949) 497–508.

[17] P. Deligne, "La conjecture de Weil : I," *Publications Mathématiques de l'IHÉS* **43** (1974) 273–307.

[18] E. Witten, "Constraints on Supersymmetry Breaking," *Nucl. Phys. B* **202** (1982) 253.

[19] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, arXiv:hep-th/9503124.

[20] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, and C. Vafa, "Geometric Singularities and Enhanced Gauge Symmetries," *Nucl. Phys.* **B481** (1996) 215–252, arXiv:hep-th/9605200.

[21] S. H. Katz and C. Vafa, "Matter from Geometry," *Nucl. Phys.* **B497** (1997) 146–154, arXiv:hep-th/9606086.

[22] S. H. Katz, A. Klemm, and C. Vafa, "Geometric engineering of quantum field theories," *Nucl. Phys. B* **497** (1997) 173–195, arXiv:hep-th/9609239.

[23] I. Volovich, "P^-adic Space-Time and String Theory," *Theor. Math. Phys.* **71** (1987) 574–576.

[24] P. G. Freund and E. Witten, "Adelic String Amplitudes," *Phys. Lett. B* **199** (1987) 191.

[25] P. H. Frampton and Y. Okada, "The P-adic String N Point Function," *Phys. Rev. Lett.* **60** (1988) 484.

[26] S. S. Gubser, J. Knaute, S. Parikh, A. Samberg, and P. Witaszczyk, "p-adic AdS/CFT," *Commun. Math. Phys.* **352** no. 3, (2017) 1019–1059, arXiv:1605.01061 [hep-th].

[27] B. Stoica, "Building Archimedean Space," arXiv:1809.01165 [hep-th].

[28] D. Ghoshal and A. Sen, "Tachyon condensation and brane descent relations in p-adic string theory," *Nucl. Phys. B* **584** (2000) 300–312, arXiv:hep-th/0003278.

[29] G. W. Moore, "Arithmetic and attractors," arXiv:hep-th/9807087.

[30] P. Candelas, X. de la Ossa, and F. Rodriguez-Villegas, "Calabi-Yau manifolds over finite fields – I," arXiv:hep-th/0012233.

[31] P. Candelas, X. de la Ossa, and F. Rodriguez Villegas, "Calabi-Yau manifolds over finite fields – II," *Fields Inst. Commun.* **38** (2013) 121–157, arXiv:hep-th/0402133.

[32] S. Kachru, R. Nally, and W. Yang, "Supersymmetric Flux Compactifications and Calabi-Yau Modularity," arXiv:2001.06022 [hep-th].

[33] Y. H. J. Lam, "Calabi-yau threefolds over finite fields and torsion in cohomologies," arXiv:2009.09951 [math.AG].

[34] Y. H. J. Lam and A. Tripathy, "Attractors are not Algebraic," arXiv:2009.12650 [math.NT].

[35] S. L. Adler, "Quaternionic Quantum Field Theory," *Commun. Math. Phys.* **104** (1986) 611.

[36] L. N. Chang, Z. Lewis, D. Minic, and T. Takeuchi, "Galois Field Quantum Mechanics," *Mod. Phys. Lett. B* **27** (2013) 1350064, arXiv:1205.4800 [quant-ph].

[37] M. Kim, "Arithmetic Gauge Theory: A Brief Introduction," *Mod. Phys. Lett. A* **33** no. 29, (2018) 1830012, arXiv:1712.07602 [math.NT].

[38] R. Schmidt, *Arithmetic Gravity and Yang-Mills Theory: An Approach to Adelic Physics via Algebraic Spaces*. PhD thesis, Munster U., 2008. arXiv:0809.3579 [hep-th].

[39] S. H. Shenker, "Another length scale in string theory?," arXiv:hep-th/9509132.

[40] M. R. Douglas, D. N. Kabat, P. Pouliot, and S. H. Shenker, "D-branes and short distances in string theory," *Nucl. Phys.* **B485** (1997) 85–127, arXiv:hep-th/9608024.

[41] J. J. Heckman, "Statistical Inference and String Theory," *Int. J. Mod. Phys. A* **30** no. 26, (2015) 1550160, arXiv:1305.3621 [hep-th].

[42] A. Bedroya, R. Brandenberger, M. Loverde, and C. Vafa, "Trans-Planckian Censorship and Inflationary Cosmology," arXiv:1909.11106 [hep-th].

[43] A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," arXiv:1909.11063 [hep-th].

[44] T. D. Brennan, F. Carta, and C. Vafa, "The String Landscape, the Swampland, and the Missing Corner," *PoS* **TASI2017** (2017) 015, arXiv:1711.00864 [hep-th].

[45] E. Palti, "The Swampland: Introduction and Review," *Fortsch. Phys.* **67** no. 6, (2019) 1900037, arXiv:1903.06239 [hep-th].

[46] J. J. Heckman, C. Lawrie, L. Lin, J. Sakstein, and G. Zoccarato, "Pixelated Dark Energy," *Fortsch. Phys.* **67** no. 11, (2019) 1900071, arXiv:1901.10489 [hep-th].

[47] D. Kadoh, "Recent progress in lattice supersymmetry: from lattice gauge theory to black holes," *PoS* **LATTICE2015** (2016) 017, arXiv:1607.01170 [hep-lat].

[48] B. S. Acharya and M. R. Douglas, "A Finite landscape?," arXiv:hep-th/0606212.

[49] A. Calderbank and P. W. Shor, "Good quantum error correcting codes exist," *Phys. Rev. A* **54** (1996) 1098, arXiv:quant-ph/9512032.

[50] A. Steane, "Multiple particle interference and quantum error correction," *Proc. Roy. Soc. Lond. A* **452** (1996) 2551, arXiv:quant-ph/9601029.

[51] A. Ashikhmin and E. Knill, "Nonbinary quantum stabilizer codes," arXiv:quant-ph/0005008.

[52] Y. I. Manin, "New Dimensions in Geometry," *Lecture Notes in Math* **1111** (1984) 59–101.

[53] A. S. Schwarz and I. Shapiro, "Supergeometry and Arithmetic Geometry," *Nucl. Phys.* **B756** (2006) 207–218, arXiv:hep-th/0605119.

[54] M. Srednicki, *Quantum field theory*. Cambridge University Press, 1, 2007.

[55] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[56] P. Dondi and H. Nicolai, "Lattice Supersymmetry," *Nuovo Cim. A* **41** (1977) 1.

[57] T. Banks and P. Windey, "Supersymmetric Lattice Theories," *Nucl. Phys. B* **198** (1982) 226–236.

[58] S. Krishna, A. Shukla, and R. Malik, "General $\mathcal{N} = 2$ supersymmetric quantum mechanical model: Supervariable approach to its off-shell nilpotent symmetries," *Annals Phys.* **351** (2014) 558–570, arXiv:1310.0262 [hep-th].

[59] A. Grothendieck, "On the de rham cohomology of algebraic varieties," *Publications Mathématiques de l'IHÉS* **29** (1966) 95–103.

[60] A. Grothendieck, "Crystals and the de Rham Cohomology of Schemes," in *Dix exposés sur la cohomologie des schémas*, no. 3 in Adv. Stud. Pure. Math., pp. 306–358. North-Holland, Amsterdam, 1968.

[61] P. Berthelot, *Cohomologie Cristalline des Schémas de Caractéristique $p > 0$*, vol. 407 of *Lecture Notes in Mathematics*. Springer-Verlag, 1974.

[62] P. Berthelot, "Géométrie rigide et cohomologie des variétés algébriques de caractéristique p," in *Introductions aux cohomologies p-adiques*, D. Barsky and P. Robba, eds., no. 23 in Mémoires de la Société Mathématique de France, pp. 7–32. Société mathématique de France, 1986. http://www.numdam.org/item/MSMF_1986_2_23__R3_0.

[63] B. Le Stum, "An introduction to rigid cohomology,". http://irma.math.unistra.fr/IMG/pdf/CoursLeStum.pdf.

[64] B. Le Stum, *Rigid Cohomology*. Cambridge Tracts in Mathematics. Cambridge University Press, 2007.

[65] E. Witten, "Supersymmetry and Morse theory," *J. Diff. Geom.* **17** no. 4, (1982) 661–692.

[66] L. Alvarez-Gaumé, "Supersymmetry and the Atiyah-Singer Index Theorem," *Commun. Math. Phys.* **90** (1983) 161.

[67] K. S. Kedlaya, "Fourier transforms and p-adic 'weil ii'," *Compositio Mathematica* **142** no. 6, (2006) 1426–1450, arXiv:math/0210149 [math.NT].

[68] E. Goncharov, "Weil Conjectures Exposition," *arXiv e-prints* (July, 2018) arXiv:1807.10812, arXiv:1807.10812 [math.AG].

[69] M. Atiyah, "Edinburgh Lectures on Geometry, Analysis and Physics," arXiv:1009.4827 [math.AG].

[70] Z. Komargodski and N. Seiberg, "Comments on the Fayet-Iliopoulos Term in Field Theory and Supergravity," *JHEP* **06** (2009) 007, arXiv:0904.1159 [hep-th].

[71] E. Witten, "Phases of N=2 theories in two-dimensions," *AMS/IP Stud. Adv. Math.* **1** (1996) 143–211, arXiv:hep-th/9301042.

[72] P. Achinger, "A Characterization of Toric Varieties in Characteristic p," *International Mathematics Research Notices* **2015** no. 16, (09, 2014) 6879–6892, arXiv:1303.5905 [math.AG].

[73] J. J. Heckman, "Lectures on Geometric Engineering," in *Graduate Summer School on the Mathematics and Physics of Hitchin Systems at the Simons Center for Geometry and Physics*. 2019. http://scgp.stonybrook.edu/archives/25133.

[74] C. Beasley, J. J. Heckman, and C. Vafa, "GUTs and Exceptional Branes in F-theory - I," *JHEP* **01** (2009) 058, arXiv:0802.3391 [hep-th].

[75] L. B. Anderson, J. J. Heckman, and S. Katz, "T-Branes and Geometry," *JHEP* **05** (2014) 080, arXiv:1310.1931 [hep-th].

[76] L. B. Anderson, J. J. Heckman, S. Katz, and L. P. Schaposnik, "T-Branes at the Limits of Geometry," *JHEP* **10** (2017) 058, arXiv:1702.06137 [hep-th].

[77] D.-E. Diaconescu, R. Donagi, R. Dijkgraaf, C. Hofman, and T. Pantev, "Geometric transitions and integrable systems," *Nucl. Phys. B* **752** (2006) 329–390, arXiv:hep-th/0506196.

[78] D.-E. Diaconescu, R. Donagi, and T. Pantev, "Geometric transitions and mixed hodge structures," *Adv. Theor. Math. Phys.* **11** no. 1, (2007) 65–89, arXiv:hep-th/0506195.

[79] D.-E. Diaconescu, R. Donagi, and T. Pantev, "Intermediate Jacobians and ADE Hitchin Systems," arXiv:hep-th/0607159.

[80] R. Donagi, "Spectral Covers," arXiv:alg-geom/9505009 [math.AG].

[81] Y. Laszlo and C. Pauly, "On the Hitchin Morphism in Positive Characteristic," arXiv:math/0008238 [math.AG].

[82] G. Laumon and B. C. Ngô, "Le lemme fondamental pour les groupes unitaires," arXiv:math/0404454 [math.AG].

[83] Ngô, B. C., "Le lemme fondamental pour les algèbres de lie," *Publications Mathématiques de l'IHÉS* **111** (2010) 1–169, arXiv:0801.0446 [math.AG].

[84] R. Donagi and M. Wijnholt, "Breaking GUT Groups in F-Theory," *Adv. Theor. Math. Phys.* **15** no. 6, (2011) 1523–1603, arXiv:0808.2223 [hep-th].

[85] S. Schafer-Nameki and T. Weigand, "F-theory and 2d $(0, 2)$ theories," *JHEP* **05** (2016) 059, arXiv:1601.02015 [hep-th].

[86] F. Apruzzi, F. Hassler, J. J. Heckman, and I. V. Melnikov, "UV Completions for Non-Critical Strings," *JHEP* **07** (2016) 045, arXiv:1602.04221 [hep-th].

[87] R. Donagi, "Heterotic / F theory duality: ICMP lecture," in *Mathematical physics. Proceedings, 12th International Congress, ICMP'97, Brisbane, Australia, July 13-19, 1997*, pp. 206–213. 1998. arXiv:hep-th/9802093.

[88] C. Vafa and E. Witten, "A Strong Coupling Test of S-Duality," *Nucl. Phys.* **B431** (1994) 3–77, arXiv:hep-th/9408074.

[89] A. Kapustin and E. Witten, "Electric-Magnetic Duality And The Geometric Langlands Program," *Commun. Num. Theor. Phys.* **1** (2007) 1–236, arXiv:hep-th/0604151.

[90] S. Lang, *Introduction to Arakelov Theory*. Springer-Verlag, 1988.

[91] R. Donagi and M. Wijnholt, "Model Building with F-Theory," *Adv. Theor. Math. Phys.* **15** no. 5, (2011) 1237–1317, arXiv:0802.2969 [hep-th].

[92] S. J. Arakelov, "Intersection Theory of Divisors on an Arithmetic Surface," *Mathematics of the USSR-Izvestiya* **8** no. 6, (1974) 1167.

[93] G. Faltings, "Intersection Theory of Divisors on an Arithmetic Surface," *Annals of Math.* **119** no. 2, (1984) 387–424.

[94] R. Schimmrigk, "Flux Vacua and Modularity," arXiv:2003.01056 [hep-th].

[95] J. López Peña and O. Lorscheid, "Mapping F_1-land: An overview of geometries over the field with one element," arXiv:0909.0069 [math.AG].

[96] J. J. Heckman and H. Verlinde, "Evidence for F(uzz) Theory," *JHEP* **01** (2011) 044, arXiv:1005.3033 [hep-th].

[97] P. A. Damianou, "On the characteristic polynomial of cartan matrices and chebyshev polynomials," arXiv:1110.6620 [math.RT].

[98] N. Nekrasov and A. S. Schwarz, "Instantons on noncommutative $\mathbb{R}^4$ and $(2,0)$ superconformal six-dimensional theory," *Commun. Math. Phys.* **198** (1998) 689–703, arXiv:hep-th/9802068.

[99] N. Seiberg and E. Witten, "String Theory and Noncommutative Geometry," *JHEP* **09** (1999) 032, arXiv:hep-th/9908142.

[100] D. Dummit and R. Foote, *Abstract Algebra*. Wiley, 2003.

[101] "Wikipedia Entry on Inverse Limit,". https://en.wikipedia.org/wiki/Inverse_limit.

[102] E. Witt, "Zyklische Körper und Algebren der Characteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p^n," *Journal für die Reine und Angewandte Mathematik* **176** (1937) 126–140.

[103] "Wikipedia Entry on Witt Vectors,". https://en.wikipedia.org/wiki/Witt_vector.

[104] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2010.

[105] R. Vakil, "Algebraic Geometry Notes,". https://math.stanford.edu/~vakil/725/course.html.

[106] J. Milne, "Lectures on Étale Cohomology,". https://www.jmilne.org/math/CourseNotes/LEC.pdf.

[107] "Wikipedia Entry on Étale Morphism,". https://en.wikipedia.org/wiki/etale_morphism.

[108] B. Conrad, "Étale Topology Notes,". http://virtualmath1.stanford.edu/~conrad/Weil2seminar/Notes/L2.pdf.

[109] Günter Tamme, *Introduction to Étale Cohomology*. Springer-Verlag, 1994.

[110] "Wikipedia Entry on Crystalline Cohomology,". https://en.wikipedia.org/wiki/Crystalline_cohomology.

[111] M. Artin, "The étale topology of schemes," *Proc. Internat. Congress Mathematicians (Moscow, 1966), Mir (1968)* 44–56.

[112] F. Zeng, *Nonlinear Codes: Representation, Constructions, Minimum Distance Computation and Decoding*. Universitat Autònoma de Barcelona,, 2014.

[113] T. Høholdt, J. van Lint, and G. Pellikaan, "Algebraic geometry codes," in *Handbook of Coding Theory, Vol. 1*, V. Pless and W. Huffman, eds., pp. 871–961. Elsevier, Netherlands, 1998.

[114] J.-L. Kim and G. L. Matthews, "Quantum error-correcting codes from algebraic curves," in *Advances in Algebraic Geometry Codes, Series on Coding Theory and Cryptology: Vol. 5*, E. Martinez-Moro, C. Munuera, and D. Ruano, eds., pp. 419–444. World Scientific, 2008.

[115] V. Savin, "Algebraic-geometric codes from vector bundles and their decoding," arXiv:0803.1096 [cs.IT].

[116] D. Gottesman, *Stabilizer Codes and Quantum Error Correction*. PhD thesis, Caltech, 1997. arXiv:quant-ph/9705052.