Speculations on Physical Discretization and Arithmetic Geometry

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Abstract

We present some speculations concerning quantum systems in which there is a discretization in the values of fields and the spacetime due to the presence of a cutoff in the target space. This can be viewed as specifying a quantum theory in which the reduced Planck constant $\hbar$ satisfies the relation $2\pi\hbar = N$ with N a positive integer greater than one. Number theoretic structures such as finite fields and schemes in characteristic p enter in a structural way, and can be packaged in the language of arithmetic geometry.

The main idea in the proposal is to view the path integral of quantum field theory as specifying a sum over rational morphisms between varieties. Introducing an action constructed from kinetic and potential energy terms then provides an algorithmic procedure for reading off correlation functions in this setting. The broad contours of this proposal appear to be in line with Swampland considerations since the evaluation of any characteristic p map automatically truncates to a finite set.

When there is an additional fibration structure with a distinguished "time coordinate" we show that the path integral formulation also comes equipped with various approximations to the standard Hilbert space of states. One approximation reduces to what might be referred to as the usual approximation in lattice quantum field theory, but including all of the data of morphisms leads to additional maps and an enlarged Hilbert space of states. The associated geometries can be interpreted as building up a physical system from quantum error correcting codes. The partial ordering of morphisms according to the degree of local maps implements a notion of UV versus IR modes, and a corresponding notion of entanglement across different scales.

The underlying structure in the étale topology also leads to a notion of momentum and winding modes for field theories in characteristic p varieties, which in some cases can be interchanged, much as in standard T-duality. We also show how some of the structure of scattering amplitudes based on rational functions defined on twistor space carries over to the characteristic p setting. Some aspects of topological field theories also carry over as well. Notions such as supersymmetry also have characteristic p analogs.

This also leads us to some physically motivated conjectures connected with this construction. Using the developed formalism, we take some first steps in analyzing the geometry associated with quantized Fayet-Iliopolous parameters. We propose a relation between supersymmetric indices and the Hasse-Weil Zeta function of schemes in characteristic p as well as a characteristic p analog of geometric engineering, including a conjectural correspondence between m-folds of ADE singularities with total space a Calabi-Yau variety, and gauge theories of ADE type.

We also show how to lift these structures to the geometry of p-adic varieties, which we use to make contact with earlier proposals for formulating physical systems over p-adic spaces in the special case where $N = p^a$ with a taken very large. In the limit where we demand convergence of the action in the p-adic topology, the target space is also of mixed characteristic. If we instead demand convergence in the sum over phase factors appearing in the path integral, convergence is enforced in the real topology, allowing us to make contact with more "standard" notions of p-adic physics where the source of the field theory is of mixed characteristic, but the target space is defined over the real numbers.

As potential physical applications, we point out that theories with eight real supercharges such as those captured by Seiberg-Witten theory have an intrinsically arithmetic structure which is closely connected with the counting of points in the corresponding arithmetic geometry. We also use this framework to revisit various proposals for constructing p-adic avatars of holography and especially the AdS/CFT correspondence. This also leads us to a notion of quantum entanglement amongst states labelled by p-adic numbers.

The appearance of limits of operator algebras provides a physically motivated extension of these considerations to the formulation of physical theories on p-adic analytic spaces. We use this to provide a new perspective on the construction of open and closed p-adic string theory over Berkovich spaces. In particular, the analytic structure allows us to retain much of the geometric flavor present in the Archimedean string setting. In a suitable large N limit consisting of multiple prime factors, we also observe the emergence of more fine-grained topological features.

A number of Appendices serve to supplement the main elements of this proposal, which range from review of some of the relevant mathematical background to additional computations which provide support for some of the proposed speculations.

Contents

Part I

Introduction and Motivation

1 Introduction

There is a seductive appeal to discretizing the laws of Nature. That being said, our best understanding of fundamental physics continues to make heavy use of continuum concepts.

In this note we argue that in some quantum systems with a cutoff, there is a natural formulation in terms of structures which appear in number theory and in particular arithmetic geometry. This algebro-geometric language provides a way to transport many features of smooth geometry to a discretized setting. Our discussion will necessarily be on the rather speculative side, but hopefully this will not distract too much from the main contours of the proposal.

To keep our analysis well-defined, we shall mainly focus on situations in which the spacetime as well as the target space for our fields are discretized in some way. This sort of situation arises in many physical situations. For example, an experimentalist may only be able to probe a system at a minimal time interval t_{min}, and moreover, the values that are recorded by their measuring device may also be limited to some finite discretized level of approximation. At a more ambitious level, one might consider formulations of quantum gravity in which there is a minimal Planck time and Planck length for measurements.

Of course, some immediate issues with studying these sorts of systems is that lattice formulations of quantum theories tend to break most spacetime symmetries (such as Lorentz symmetry), and extreme fine-tuning is often required to recover these structures at long distances.[1] Similar issues are often present in non-commutative deformations of spacetime as well as matrix model approaches to quantum gravity. In the arithmetic context, however, there are analogs of the Lorentz group which can be maintained even in the discretized setting.

Our operating assumption will be that once we discretize the target space and spacetime, there is a natural sense in which the number of quanta which can be packed into any region of the target space is discretized. For example, in the case of a bosonic scalar field theory, we can interpret this in terms of the standard prescription for computing operator correlation functions via the path integral through expressions such as:

$$\langle \widehat{\mathcal{O}}_1 ... \widehat{\mathcal{O}}_m \rangle = \frac{\sum\limits_{\phi} [d\phi] \exp\left(iS[\phi]/\hbar\right) \mathcal{O}_1 ... \mathcal{O}_m}{\sum\limits_{\phi} [d\phi] \exp\left(iS[\phi]/\hbar\right)}, \tag{1.1}$$

but in which the fields ϕ range over a discrete set such as the integers, and the parameter $\hbar$ satisfies the condition:

$$\hbar = \frac{N}{2\pi}, \tag{1.2}$$

with $N > 1$ a positive integer. In this setting, evaluating the action on any field configuration

[1] It is also unclear how lattice formulations can be recoupled to quantum gravity, where the spacetime itself is expected to fluctuate.

results in $S[\phi]$ an integer, but in which only its value modulo N actually matters. One of our aims will be to show how to extend this sort of observation to the standard set of fields encountered in quantum field theory, including fermions, gauge fields, and even gravitons. We will also argue that supersymmetry still makes sense.

As one might expect, additional features become manifest if we restrict to the special case where $2\pi\hbar = p$ a prime number. When we do so, we can borrow much of the apparatus of algebraic geometry in characteristic p to formulate and study the resulting physical systems.[2] Again by way of example, we will show that bosonic scalars of the discretized field theory can be viewed as specifying a map between schemes defined in characteristic p:

$$\phi : X_{\text{spacetime}} \to Y_{\text{target}}. \tag{1.3}$$

Of course, the sense in which the "spacetime" of the field theory makes sense in characteristic p is that it is a suitably discretized space, and we can equip it with a topology, and non-trivial sheaves and maps to other spaces. See figure 1 for a depiction.

We use this perspective to propose how various field theories can be defined in the characteristic p setting. It is also clear that some cherished physical structures such as a notion of distance as defined by a metric will necessarily fall by the wayside in the characteristic p setting. There are, however, close analogs which retain much of the physical flavor present in Riemannian geometry in characteristic zero. For example, we can consider the space of symmetric bilinear forms specifying maps $T^*X \otimes T^*X \to \mathbb{F}_p$. This is the characteristic p analog of the graviton. Indeed, our formulation will be flexible enough to demand that we only work with structures invariant under suitable coordinate redefinitions, as captured by morphisms between schemes.

Another aim of our analysis will be to study the impact on effective field theories generated in this way. We find that the the information contained in the spectrum of higher dimension operators truncates, at least when evaluating the effective action on specific field configurations. This in some sense follows from the fact that in modular arithmetic, we have Fermat's little theorem, which tell us that for integers $m \in \mathbb{Z}$, reduction modulo p a prime always satisfies $m^p = m \bmod p$. Applied to a power series expansion in a physical field, this automatically leads to a truncated effective action. This is in line with some Swampland considerations such as [1–3] which suggest the appearance of correlated Wilson coefficients in any quantum field theory coupled to higher dimension operators. That being said, our proposal will involve working with the space of all possible rational morphisms between varieties, and as a consequence, there will still be a formally infinite number of independent Wilson coefficients, even though the effective potential can only ever attain a finite number of possibilities. The formalism also seems to be flexible enough to accommodate much of the mathematical structure in the study of scattering amplitudes, especially in terms of potential formulations in twistor space.

[2] We review some aspects of arithmetic in characteristic p later on.

More conceptually, we can use this path integral formalism to specify an implicit notion of physical states. Even though we are working mod p so that no total ordering is available, there is a notion of past and future with respect to a local time coordinate. We can achieve this because our morphisms are locally presented as Laurent series of finite degree, and negative degree terms are associated with modes in the past, and positive degree terms are associated with modes in the future. Provided the spacetime X admits a fibration of the form $X_s \to X \to X_t$ over a dimension one variety X_t which we refer to as the "time direction", then rational morphisms involving just the spatial support serve to construct states $|\Phi : X_s \dashrightarrow Y\rangle$ in a "big Hilbert space" $\mathcal{H}_{\text{big}}$. Including additional polar terms associated with a local coordinate in X_t enlarges this space further to a "BIG Hilbert space" which we refer to as $\mathcal{H}_{\text{BIG}}$. Of course, one can also consider the explicit evaluation maps for these morphisms, and this leads to a "small Hilbert space" $\mathcal{H}_{\text{small}}$.

The appearance of rational morphisms between varieties provides a natural starting point for constructing both classical and quantum error correcting codes. In this interpretation, the path integral sums over possible codes, and introduces a preferential complex phase, as dictated by the choice of action. In this sense, the entire edifice of "physics in characteristic $p > 0$" can be recast in information-theoretic terms. The use of quantum error correcting codes in quantum gravity has been a topic of some interest in the holographic quantum gravity literature (see e.g., [4–7]), and it would of course be natural to make closer contact with these considerations.

In line with these considerations, we also argue that even in characteristic p we can organize the physical degrees of freedom according to a hierarchy of scales, and that "coarse graining" naturally leads to a notion of entanglement across scales. Locally, we can treat morphisms $X_s \dashrightarrow Y$ as polynomials, and higher degree modes can be interpreted as specifying "UV data" while lower degree terms specify "IR data". This leads to tree-like structures which are reminiscent of related constructs which have appeared in the study of holographic tensor network models (see e.g., [8–10]) as well as the p-adic AdS/CFT correspondence [11–13].

At a more practical level, the fact that we can approximate morphisms in terms of local coordinate expansions means that we still have a mode expansion available which allows us to carry out explicit evaluations of correlation functions. To illustrate we consider some toy models and explicitly work out the corresponding path integral manipulations. An interesting feature of this setup is that some loop corrections automatically vanish in this setting, leading to suppressed contributions to various bubble diagrams. That being said, loop corrections are in general present, leading to a non-trivial structure for such systems.

Other celebrated notions from the study of field theory, including symmetries and currents, as well as topological actions appear to have characteristic p versions. In the case of our topological actions, this requires some further speculations for how cobordism theory might operator in finite as well as mixed characteristic.

We envision applying these sorts of considerations to the study of quantum gravitational

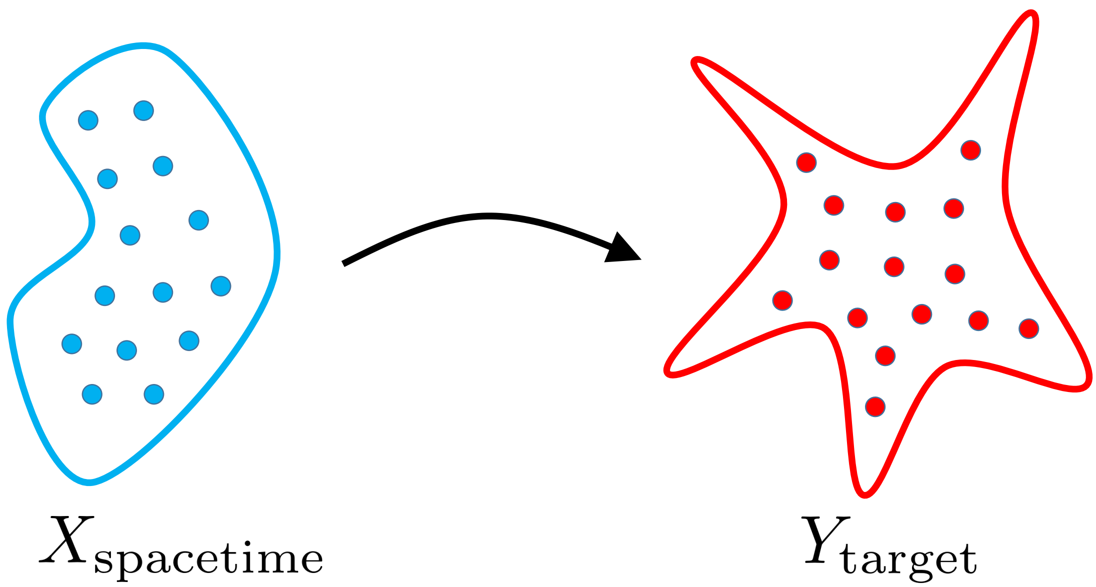

Figure 1: Depiction of a map between two varieties in characteristic p. Each dot indicates a point of the corresponding discretized geometry. These maps can be viewed as field configurations in a sigma model, with $X_{\text{spacetime}}$ the spacetime and Y_{target} the target space.

systems, including (as already mentioned) in the study of quantum error correction. As another example, we consider the analog of supersymmetric field theory with a quantized Fayet-Iliopoulos (FI) parameter. There are hints from supergravity (see e.g., [14–17]) which indicate that FI parameters in 4D systems may exist, provided they are quantized in units of $2M_{\text{pl}}^2$ with $M_{\text{pl}}^2 = (8\pi G)^{-1}$ the reduced Planck mass squared, with G the 4D Newton's constant. Assuming this is possible, we study the characteristic p analog of supersymmetric vacua in the presence of an FI parameter. We also find that some notions of the resulting symplectic geometry carry over, giving us a notion of toric varieties in characteristic p. That being said, the fact that there is no notion of "big or small" in characteristic p means that there is little sense in which we can reach a semi-classical geometry in this case. We do, however, find that there is a suitable notion of quasi-locality, as specified by the Grothendieck topology. We note that the introduction of physical topoi has been discussed for example in references [18–23] but we leave to future work any attempt to align with the considerations found therein.

We believe that the present formulation also sheds light on some structures which appear in number theory, although we leave a study of this for future work. For example, a well known quantity is the Hasse-Weil Zeta function [24, 25] for a variety V defined over a finite field $\mathbb{F}_q$ in characteristic p. A classic question in this subject is to compute the number of points as defined over a finite field $\mathbb{F}_{q^n}$. This is all packaged in terms of the Hasse-Weil Zeta function:

$$Z_{V,q}(z) = \exp\left(\sum_{n\geq 1} \#V(\mathbb{F}_{q^n})\frac{z^n}{n}\right). \tag{1.4}$$

Given the setup just explained, it is tempting to view this data as being specified by a supersymmetric index for a physical system [26], but now in characteristic p:

$$\sum_{n\geq 1}\mathrm{Tr}_n\left((-1)^{\mathbf{F}}z^n\right) = \log Z_{V,q}(z). \tag{1.5}$$

The appearance of an algebraic formulation for our physical system also enables us to analyze systems of direct relevance in string compactification. Typically, the string compactification geometry is treated as some large volume approximation to a more accurate quantum corrected system. Since, however, our entire formulation is algebraic, classic constructions such as geometric engineering [27–30] have characteristic p analogs, and allow us to formulate a conjectural correspondence between m-folds of ADE singularities with total space a Calabi-Yau variety, and gauge theories of ADE type.

We also consider the mixed characteristic case, i.e., where we work over a p-adic space. The appearance of p-adic numbers occurs in two natural ways in this setting. First of all, we can consider $N = p^a$ a prime power, and in the limit $a \to \infty$, our path integral formalism is well-approximated by a field theory defined by morphisms between varieties over the p-adic numbers. A related comment is that there is a formal lifting procedure to go from characteristic p geometries to p-adic geometries which provides a systematic lift of our setup. We comment that in our case, the emphasis (at least for characteristic p spaces) is really on rational morphisms $\phi : X \dashrightarrow Y$ between p-adic schemes rather than the more typical situation encountered in the p-adic physics literature which involves maps to geometries defined over the real numbers. We can, however, pass to this special case by composing these morphisms with character maps, and in this case we observe a fit with "standard" notions from p-adic physics.

The treatment of the path integral in the p-adic setting provides us with our first encounter with a p-adic differential equation. In particular, this also provides us with a way to make sense of period integrals such as those which appear in the study of Seiberg-Witten theory and the B-model of topological string theory. Here, the operating theme is that so long as a formal power series expansion is available (with a suitable radius of convergence), then there is a notion of solving these differential equations.

Even more tantalizing is that there is a notion of "monodromy" in the p-adic setting, and this tracks well with the corresponding notions present in the complex analytic setting. This in particular means that some of the crucial properties of monodromy present in the analysis of massless states which appear in Seiberg-Witten theory translate over to the arithmetic setting as well. In fact, these considerations provide a direct connection between the arithmetic properties of the central charge of the corresponding massless state and the arithmetic of the Seiberg-Witten curve, in particular its Zeta function.

Taking the large N limit also provides a route to recovering continuum notions such as more refined topological spaces. Taking this limit at the level of the action and fields actually produces, for $N = p^a$ with p a prime, a completion inside the p-adic numbers. If we instead take a limit as obtained on the "phase factors" of the path integral valued in $S^1 \subset \mathbb{C}^*$, we instead see a completion available in the real numbers. This provides a general route for recovering a continuum limit from our general discretized considerations.

Indeed, we also consider the resulting structure of operator algebras in this context, and this motivates us to take seriously various limits of operator algebras in the large $N = p^a$ limit. Such limits provide a physical motivation for "filling" in the p-adic topology with additional points, and naturally motivates the appearance of additional topological structure. The first step in this direction leads us to the appearance of Tate's rigid analytic spaces [31], though we find that further physical demands such as analyticity of operator correlation functions require refining this to a path connected space, as is obtained in the p-adic analytic spaces of Berkovich [32], and generalizations thereof. In fact, from this starting point we can give a proposal for a analytification of the p-adic string which we refer to as the "Berkovich string," presenting both closed and open string variations. An important feature of working with p-adic analytic worldsheets is that much of the flavor of holomorphic geometry used in the Archimedean string admits an analogous treatment in the non-Archimedean p-adic analytic setting.

In the more general case where $N = p_1^{a_1} ... p_m^{a_m}$ is a product of primes, the arithmetic perspective also provides helpful hints for how to proceed. In this case, it is fruitful to view our geometry as fibered over the "affine line" $\operatorname{Spec} \mathbb{Z}$. By performing a path integral over this bigger geometry, we can uniformly treat all primes at once. Viewed in this way, fixing a particular value of N amounts to a semi-classical approximation. Localization near a given prime divisor p of N then leads to a similar arithmetic interpretation for a general integer N.

Lastly, let us mention that the notion of looking for connections between number theory and physics is certainly not new to us. Indeed, many intriguing connections between p-adic numbers and strings have been appreciated for some time in both early work such as references [33–41] as well as in more recent work such as [11–13, 42–64]. The subject of p-adic strings also shows up in some approaches to studying tachyon condensation [65]. Connections with certain arithmetic structures which appear in string theory have also been noted (see e.g., [66] for an example of this sort). Relations between modular forms as they appear in string theory and arithmetic questions have been considered in [67–78]. It has also been appreciated that the calculation of period integrals for some Calabi-Yau threefolds is amenable to techniques from number theory [79–86]. Some proposals for quantum mechanics with different algebras include [87,88]. We were also inspired by the arithmetic path integrals for certain topological field theories appearing in reference [89–92]. Perhaps closest in spirit to the present considerations is the work of reference [93] which sets up much of the necessary mathematical formalism in a setting close to the physical considerations explored here.

That being said, we have not tried to fully reconcile our present perspective with these considerations (with the notable exception of reference [93]) but it would be interesting to try. Indeed, our underlying motivation is somewhat different. First and foremost, our interest in the appearance of these structures is motivated by its potential use in studying physical, in principle experimentally accessible systems. But of course, we hope that this will lead to a two-way development, with physical notions helping to inform some questions in arithmetic geometry, and conversely, that such number theoretic analogs of geometry can help in the search for a fundamental formulation of physics.

The rest of this note is organized as follows. The general themes are organized according to several parts. In part I (which includes the Introduction) we give our general motivation and underlying philosophy. We begin in section 3 by discussing the sense in which discretized systems allow for different choices of $\hbar$.

In part II we turn to the formal development of physical systems in characteristic $p > 0$. In section 4 we begin with our first example, studying a discretized bosonic field. We generalize this construction in several ways, eventually arriving at a more geometric formulation amenable to study via methods in arithmetic geometry. Some aspects of the Hilbert space associated to these systems are discussed in section 5. We next show in section 6 that these structures can be interpreted as specifying a class of quantum error correcting codes. Section 7 shows that there is a natural "scale entanglement" present in the Hilbert space of state built from spatial morphisms. There is a resulting tree-like structure which is reminiscent of observations made in the context of recent holographic studies, including the p-adic AdS/CFT correspondence, a feature we return to later on. We study mode expansions in bosonic field theories in section 8, as well as some "classical" aspects of Green's functions in characteristic p in section 9. Section 10 studies symmetries and currents, and in section 11 we give a proposal for how to build topological actions. In section 12 we study physical twistors in characteristic p. Section 13 discusses the generalization to fermionic degrees of freedom, including a sketch of supersymmetric quantum mechanics in characteristic p. We also speculate on a physical interpretation of the Hasse-Weil Zeta function. Section 14 analyzes some aspects of discretized FI parameters. In section 15 we present a proposal for geometric engineering in characteristic p.

In part III we turn to the case of more general ground fields and ground rings. This includes working over fields in mixed characteristic, the main example being various p-adic spaces, as well as further fibration over $\operatorname{Spec}\mathbb{Z}$. Section 16 discusses the extension of this analysis to p-adic numbers, where we also make contact with the appearance of p-adic differential equations, both in the classical and quantum setting. Section 17 considers a specific application of these considerations, drawing out the arithmetic structure of Seiberg-Witten theory and its connection to the Zeta function of an arithmetic curve. In section 18 we discuss some holographic structures which are present in physics over the p-adics, and in section 19 we make contact with more "standard treatments" in the literature. Motivated by the need to obtain suitable limits of operator algebras, in section 20 we consider the further generalization to p-adic analytic spaces. In section 21 we consider more general arithmetic systems.

In part IV we wrap up the main body of the text. We present our conclusions in section 22. Acknowledgements are contained in section 23. Some comments on the version history of this document are included in section 24.

In part V we provide a number of Appendices. We have ordered them according to where they roughly appear in the ordering presented in the main body of the text. The Appendices include some additional developments which range from original statements to review of various topics which may be unfamiliar to the "typical" reader. In Appendix A we analyze some 1D quantum systems in characteristic p. In Appendix B we explain how to pass from finite differences of the sort which appear in lattice systems to the Hasse derivatives which figure prominently in characteristic p geometries. Appendix C reviews some aspects of finite fields, and in Appendix D discusses some aspects of geometry in characteristic p. In Appendix E we present a brief review of Grothendieck topologies. Some aspects of classical and quantum error correcting codes obtained via geometry in characteristic p are discussed in Appendix F. Appendix G treats the evaluation of the partition function for a free field. Appendix H presents an alternative way to define an action principle in finite characteristic which directly references a preferred mode expansion basis. It also explains some of the difficulties in extending this to more general geometries. A brief discussion of the étale fundamental group is given in Appendix J. We discuss some aspects of real and complex twistors in Appendix L. In Appendix M we present an "alternative" supersymmetric quantum mechanics system in which we impose a different rule for Frobenius conjugation on Grassmann fields. In Appendix K we list a few examples of Zeta functions. Appendix N presents some evidence that FI parameters can be quantized in string constructions. In Appendix I we review the construction of inverse limits and in Appendix P we review some aspects of Witt vectors. Appendix O provides a brief introduction to the p-adic numbers. We review the convergence properties of the p-adic exponential function in Appendix Q. In Appendix R we review some aspects of the p-adic logarithm and its generalizations. Appendix S discusses some aspects of ramification theory for algebraic and local fields, and in Appendix T we discuss an explicit example of monodromy in the context of ℓ-adic cohomology of an elliptic curve of the sort which appears in Seiberg-Witten theory. In Appendix U we review some aspects of Berkovich spaces, and in Appendix V we briefly discuss tropicalization maps. Appendix W includes some analysis of p-adic integrals which appear in the computation of various p-adic string amplitudes.

2 Disclaimers

As a general disclaimer, the nature of this work is, as already mentioned, quite speculative. It is speculative in terms of both the physics and mathematics which is presented, and especially on the potential connections between these themes. With that stated, we have attempted to tie together various threads to produce a coherent picture.

Additionally, some of the material presented (especially some of the Appendices) is also rather standard, and there are many well-known accounts. In such cases, we have attempted to closely follow treatments which we personally found helpful and pedagogical. Nevertheless, we have attempted to synthesize the various treatments to suit our particular needs.

Some readers may find this combination of speculations interspersed with well-known facts repellant. We hope, however, that some will find parts of it helpful in their own investigations.

With these disclaimers stated, we now proceed.

3 Discretized Target Spaces

In this section we explore some of the consequences of discretizing a target space. In subsequent sections we shall refine this analysis, showing how additional structure can be maintained by working with geometry in characteristic p.

Our eventual proposal will be to construct a path integral by working with rational morphisms between characteristic p varieties $\phi : X \dashrightarrow Y$ of the form:

$$\int_{\Phi_i}^{\Phi_f} [d\phi] \ \exp(iS[\phi]/\hbar) \equiv \sum_{\substack{\phi: X \dashrightarrow Y \\ \phi(t_f) = \Phi_f \\ \phi(t_i) = \Phi_i}} \exp\left(\frac{2\pi i}{p} S[\phi]\right). \tag{3.1}$$

where X refers to the "spacetime" and Y to the target space. Locally, each such morphism can be presented as a polynomial which includes possible finite degree meromorphic terms. In this sense, the sum over paths is just a discrete sum over the space of possible morphisms, and is thus "better behaved" than the generic situation one typically encounters in the standard path integral.

This is to be contrasted with the first attempt one might make in specifying such a path integral as obtained by discretizing the target space, as well as the source. Indeed, if one considers geometries with a finite number of points, the space of all point set mappings between such geometries is necessarily finite (one simply has to specify possible values for each point set). This is essentially a lattice approximation, and with it one encounters the usual pathologies in discretizing continuum physics. Another general worry is that if one ever wishes to couple a quantum field theory to gravity, using a fixed lattice becomes quite awkward because one expects spacetime to fluctuate anyway. We will ultimately need to abandon this way of thinking about physics in discretized target spaces but it is nevertheless instructive as a "cautionary morality tale".

So, in order to motivate the form of our path integral, we shall first proceed in the most straightforward way and ask about the consequences of discretizing the target space (and source) of a quantum field theory. To rectify these issues we will indeed need to pass over to a formulation of physics in characteristic p specified in terms of structures which appear in arithmetic geometry.

To get there, we shall first proceed by motivating the reasons for entertaining discretization at all. We emphasize that some of the first attempts mentioned in this section will need to be revisited in subsequent sections. The core idea which we will hold on to is that we can conveniently summarize discretization on a target space as setting a convention in which fields take values on the integers and the reduced Planck constant is set to:

$$\hbar = \frac{N}{2\pi}. \tag{3.2}$$

We first motivate this discretization by way of a few examples. First, we consider a point particle, then a field on a 2D spacetime. We then make some brief philosophical comments. The aim of section 4 will be to formalize some of these features. Some additional details on the lattice approximation for 1D point particles in characteristic p are presented in Appendix A. For additional details on mode expansions for quantum fields in characteristic p, see section 8.

3.1 Point Particle

As a first example, suppose we have a particle moving in one spatial dimension. Classically, we can visualize this by introducing a function of time $Y(T)$, indicating the position of our particle. A common situation is an action of the form:

$$S[\Phi] = \int L dT = \int dT \ \left(\alpha m \left(\partial_T \Phi\right)^2 - \mathcal{V}(\Phi)\right), \tag{3.3}$$

where α is an integer and m is proportional to the mass of the particle. Note that we have not canonically normalized the fields. This will be important when we turn to the discretization of our system. Here, $\mathcal{V}(\Phi)$ denotes a potential energy and in what follows we shall assume that this is always taken to be a polynomial in the field Φ. In practice, one often expands a potential energy density about some background value of Φ, say Φ_0, and then analyzes the leading order terms of such an expansion. In this sense, we expect to get a "good approximation" by just dealing with polynomials of possibly very high degree. Indeed, one expects that in a theory of quantum gravity some higher order terms may actually capture strictly redundant information [3].

Quantum mechanically, we can use this as a starting point for the path integral. For example, we are instructed to sum over possible choices of functions $\Phi(T)$, each weighted by a factor $\exp(iS[\Phi]/\hbar)$. Correlation functions involving operators $\widehat{\mathcal{O}}$ built from the Φ's are obtained in the usual way by the formal relation (we leave time-ordering implicit):

$$\left\langle \Phi_f \left| \widehat{\mathcal{O}}(T_m)...\widehat{\mathcal{O}}(T_1) \right| \Phi_i \right\rangle \equiv \frac{\int\limits_{\Phi_i}^{\Phi_f} [d\Phi] \ \exp(iS[\Phi]/\hbar)\, \mathcal{O}(T_m)...\mathcal{O}(T_1)}{\int\limits_{\Phi_i}^{\Phi_f} [d\Phi] \ \exp(iS[\Phi]/\hbar)} \tag{3.4}$$

For brevity, in what follows, we shall leave the initial and final values of the field configurations implicit.

It is natural consider possible discretizations of the above system. For example, if we consider a particle which can only occupy points on a spatial lattice, there is a minimal spacing for values of the field. Doing so, however, introduces fresh complications. For example, if we simply posit $\Phi(T)$ takes values over the integers, then our notion of a time derivative ceases to make sense. There is a workaround which is available in systems where we are also only able to make measurements at discretized time steps. For example, an observer may be limited in how frequently they can actually measure the response of the system. In a Hamiltonian evolution of a given state such as:

$$|\Psi(T)\rangle = \exp(-i\widehat{H}T/\hbar)\,|\Psi(0)\rangle\,, \tag{3.5}$$

it may be that there is a minimal time resolution, so we can only ever access discretized time steps.

Discretization in the target space and the time direction suggests a way to proceed. First, we introduce a minimal step size for the field. Additionally, we introduce a minimal time step by which a particle can actually change. Changes in the energy are then also discretized. Making these changes amounts to the lattice approximation:

$$\begin{aligned}
T &\mapsto \tau_{\text{time}} t && (3.6)\\
\Phi(T) &\mapsto \ell_{\text{target}}\phi(t) && (3.7)\\
\partial_T\Phi(T) &\mapsto \frac{\phi(t+1)-\phi(t)}{\tau_{\text{time}}} && (3.8)\\
\int dT &\mapsto \sum_t \tau_{\text{time}}, && (3.9)\\
\mathcal{V}(\phi) &\mapsto \frac{m\ell_{\text{target}}^2}{\tau_{\text{time}}^2}V(\phi) && (3.10)
\end{aligned}$$

where we now assume $t, \phi(t), \phi(y) \in \mathbb{Z}$. Returning to the form of our action, we now have:

$$S[\phi] = \sum_t \frac{m\ell_{\text{target}}^2}{\tau_{\text{time}}}\left(\alpha\left(\phi(t+1)-\phi(t)\right)^2 - V(\phi)\right). \tag{3.11}$$

Evaluating correlation fuctions now proceeds just as in the ordinary path integral. For example, the integration over all paths is now replaced by discretized sums:

$$\int_{\Phi_i}^{\Phi_f}[d\Phi] \mapsto \sum_{\phi(t_f)}...\sum_{\phi(t_i)}\delta(\phi(t_f)=\phi_f)\delta(\phi(t_i)=\phi_i) \tag{3.12}$$

where here, we have specified an initial and final field configuration. We have also dropped dimensionful factors associated with the path integral measure, since we have already now passed to the discretized setting.

The main thing we wish to explore is what happens when the dimensionless ratio involving these length scales and the Planck constant is held fixed:

$$\frac{1}{\hbar}\frac{m\ell_{\text{target}}^2}{\tau_{\text{time}}} = \frac{2\pi}{N}, \tag{3.13}$$

with N an integer. We could, of course, have jumped straight to this form of the phase factor in the path integral by working in natural units with all lengths and time steps set to one. In that case, we could assert:

$$\hbar = \frac{N}{2\pi}. \tag{3.14}$$

It is customary to work in natural units where $\hbar = c = 1$, but a priori we can consider more general choices, and they have no impact on the physics. Indeed, the classical limit is typically associated with the limit $\hbar \to 0$. Here, we are considering the opposite regime where all behavior is highly quantum, and so we have chosen to emphasize this by absorbing all these changes into the choice of the reduced Planck constant. In any event, the path integral is now weighted by factors of the form:

$$\exp(iS[\phi]/\hbar) = \exp\left(\frac{2\pi i}{N}\sum_t \left(\alpha\left(\phi(t+1) - \phi(t)\right)^2 - V(\phi)\right)\right). \tag{3.15}$$

This is where we encounter our first surprise. In these units, we observe that if all quantities in our system are discretized integers, then the only contributions we actually care about are obtained modulo N. Indeed, this is just because $\exp(2\pi i) = 1$.

One might also ask about observables in this sort of system. One class of operators which respect the observed mod N structure is given by "vertex operators" of the form:

$$U(t) = \exp(2\pi i\phi(t)/N). \tag{3.16}$$

Compared with our discussion of path integrals given above, the only difference is that now, we need not integrate over all paths, just their mod N residues. In the original continuum theory, these operators are mildly non-local, arising from expressions such as:

$$\exp\left(i\int_{t-\varepsilon}^{t+\varepsilon}\frac{dt'}{\tau_{\min}}\frac{\Phi(t')}{\ell_{\min}}\right), \tag{3.17}$$

for ε a small number. This amounts to a small amount of "smearing" in the original continuum theory.

Another comment has to do with the domain of the time coordinate. Assuming that the Hamiltonian has integer eigenvalues, we observe that the time evolution operator:

$$\exp\left(-\frac{i}{\hbar}\widehat{H}t\right) = \exp\left(-\frac{2\pi i}{N}\widehat{H}t\right) \tag{3.18}$$

repeats after at most N time steps.

3.2 2D Example

In the previous subsection we introduced a first example of a discretized system, observing the appearance of a natural mod N structure in the resulting path integral. We now present a generalization of this to the case of a 2D field theory. We focus on the case of a 2D non-linear sigma model of the sort one encounters in the study of string theory. In this case, the spacetime of the field theory consists of the worldsheet of the string. We focus on a lattice approximation to flat space $\mathbb{R}^{1,1}$ and consider the Polyakov action:

$$S[\Phi] = \frac{1}{4\pi\alpha'} \int d^2\sigma \; G_{AB}(\Phi) h^{ab} \partial_a \Phi^A \partial_b \Phi^B. \tag{3.19}$$

Here, α' has dimensions of length squared. This, of course, is the starting point for understanding perturbative strings moving in a target space with metric $G_{AB}(\Phi)$. It has been appreciated for some time that the minimal length scale in string theory is not set by $\sqrt{\alpha'}$, but can be far smaller, and involves the string coupling g_{string} as well [94, 95]. With this in mind, we explore the consequences of assuming that there is a minimal length scale which can be probed by our string. Much as in our discussion of the point particle, we first consider a rescaled version of the fields, writing:

$$\Phi \mapsto \ell_{\text{target}} \phi, \tag{3.20}$$

so that the ϕ's are valued in the integers. By the same token, we also replace all derivatives by lattice derivatives, with the worldsheet specified by points on the two-dimensional lattice $\mathbb{Z} \times \mathbb{Z}$. In this case, observe that since we are in two dimensions, the rescaling of the measure factor from the worldsheet integral cancels the rescaling of the lattice derivatives.

We would like to understand what happens when the dimensionless ratio involving the target space length scale and the string scale and the Planck constant is taken to be fixed as:

$$\frac{1}{\hbar} \frac{\ell_{\text{target}}^2}{4\pi\alpha'} = \frac{2\pi}{N}, \tag{3.21}$$

with N an integer. Much as in the case of the point particle example, we could have jumped straight to this form of the phase factor by working in natural units with all lengths and time steps fixed to one and setting:

$$\hbar = \frac{N}{2\pi}. \tag{3.22}$$

In the context of string theory, taking the large N limit means the string tension passes to zero. This is clearly far away from the realm of classical geometry.

Pressing on, most of our discussion of this 2D example proceeds as in the 1D case. We can again also speak of vertex operators such as:

$$U(\sigma^a) = \exp(2\pi i k_A \phi^A(\sigma^a)/N). \tag{3.23}$$

Again, in the continuum theory this sort of expression comes about from a mildly non-local operator with some small amount of smearing, as per our discussion below equation (3.17). One can also entertain integer valued operators as well, and the prescription for calculating correlation functions is essentially the standard one for the path integral, just with a new domain of summation / integration.

3.3 Philosophical Comments

By now, the general procedure should be clear, at least for field theories specified by scalars. We discretize the target space, and also the spacetime of the field theory.[3] The case of two dimensions is a bit special in this regard, because the actual lattice spacing of the spacetime drops out from our expression for $\hbar$. In more general systems with non-trivial operator scaling dimensions, similar considerations would likely also apply. This motivates us to study systems in which the values of fields are restricted to integers, with the reduced Planck constant set to the value:

$$\hbar = \frac{N}{2\pi}. \tag{3.24}$$

At a conceptual level, introducing this sort of discretization is appealing for a number of reasons. As we have already mentioned, there is a sense in which any measurement by an observer already comes in "quantized units." Indeed, there is a strict difference between the real numbers and those which are actually computable (see e.g., [96]). An additional comment is that fundamental physics makes reference to quantities such as a Planck time and Planck length. All of these signal some (perhaps ineffable) basic intuition that there is a minimal unit of measurement. Note also that the class of operators which naturally enter in this setting include some mild amount of non-locality, such as:

$$\exp\left(i \int\limits_{\varepsilon_D} \frac{d^D x}{\ell_{\min}^D} \frac{\Phi(x)}{\Lambda^\Delta} \right) \tag{3.25}$$

where $\ell_{\min}$ refers to a minimal length scale, and Λ is a mass scale, and Δ is the engineering dimension for a field Φ. Here, the integral takes place over a small region ε_D, indicating a mild amount of smearing / averaging. This is also in line with the expectation that there are limits to statistical inference in quantum gravity [97].

Discretizing parameters is also natural, especially in systems where moduli are stabilized. Arithmetic properties of stabilized moduli have been discussed in [66, 72, 74, 76–78, 86], and

[3]Note that one can also consider various hybrid setups, where the onset of discretization appears first in some extra dimensions, and only eventually in the actual spacetime for macroscopic observers.

so one can view our attempts to discretize the target space as very much in line with Planck scale moduli stabilization.

We also remark that in some cases, this intuition has recently been sharpened in the context of the Swampland conjectures (see e.g., [1, 2, 98, 99] and references [100–103] for reviews). One of the recurring themes in this line of research is to explore the impact of the Planck scale on long distance physics, particularly low energy effective field theories. Naive extrapolation of an effective field theory is expected to produce various pathologies, and one potential way around this is to discretize various physical structures (see e.g., [104]).

On the other hand, there are also well-known drawbacks to discretization. For one, introducing an explicit lattice cutoff immediately destroys Lorentz invariance of the system. In lattice field theory, it is common to fine-tune all parameters so that Lorentz invariance is recovered at long distances. Such an option may not be available here since we are also discretizing the parameters of the system. Another difficulty is that the proper treatment of fermions, let alone supersymmetry is rife with technical (though not insurmountable) difficulties. See for example, [105] for some recent discussion on these points. Along these lines, any notion of quantum gravity on a lattice is again potentially quite problematic since the lattice itself would need to fluctuate. An additional concern is that one of the powerful probes of quantum locality comes from analyticity of the S-matrix. Much of the power of results in scattering amplitudes comes from the fact that the S-matrix can be analytically continued to "unobservable" large and complex values of momenta. Sacrificing this in the name of discretization would be a pity.

We propose to balance these competing considerations using the geometry of numbers.[4]

[4] In the poetic sense, not the strict sense of Minkowski.

Part II

Physics in Characteristic $p > 0$

4 Discretization in Characteristic $p > 0$

In the previous section we presented an intriguing observation that some discretized systems have close contact with some crude features of arithmetic modulo N. We also saw, however, that a direct lattice approximation produces some potentially unpleasant features, particularly if we wish to maintain contact with analytic structures which appear so central to many aspects of fundamental physics. The main idea we develop here will be to consider an alternative interpretation of such discretized systems in which we leverage the "analytic structure" present in arithmetic geometry, namely algebraic geometry in characteristic $p > 0$.

With this in mind, in this section we confine our attention to the special case where $N = p$ is an odd prime number. Much of what we develop also works (with suitable amendments) in characteristic 2, but we do not discuss this special case in what follows. The main issue we need to develop is a suitable notion of a path integral, as defined by an action principle. This will require us to provide a notion of:

$$\int_{\Phi_i}^{\Phi_f} [d\phi] \, \exp(iS[\phi]/\hbar) \equiv \sum_{\substack{\phi: X \dashrightarrow Y \\ \phi(t_f)=\Phi_f \\ \phi(t_i)=\Phi_i}} \exp\left(\frac{2\pi i}{p} S[\phi]\right), \tag{4.1}$$

as well as insertions of operators, as in our discussion around equation (3.4):

$$\langle \widehat{\mathcal{O}}_1 ... \widehat{\mathcal{O}}_m \rangle \equiv \frac{\Phi_i \sum\limits_{\substack{\phi: X \dashrightarrow Y \\ \phi(t_f)=\Phi_f}} \exp\left(\frac{2\pi i}{p} S[\phi]\right) \mathcal{O}_1 ... \mathcal{O}_m}{\Phi_i \sum\limits_{\substack{\phi: X \dashrightarrow Y \\ \phi(t_f)=\Phi_f}} \exp\left(\frac{2\pi i}{p} S[\phi]\right)}, \tag{4.2}$$

where the operators $\mathcal{O}_j$ built out of the fields are viewed as taking values in the character group of a finite field.

We interpret this in the following subsections in increasing levels of abstraction, but the main idea will be to view it as a sum over all possible morphisms $\phi : X \dashrightarrow Y$.[5] The use of the dashed arrow is to remind us that we allow poles along marked subspaces in X. We

[5] Here, our main requirement is that we can present our map in terms of a polynomial in local coordinates. For affine varieties X and Y given respectively by closed (in the Zariski topology) subsets of affine spaces $\mathbb{A}^n$ and $\mathbb{A}^m$, a morphism $\phi : X \to Y$ is given by the restriction of the appropriate polynomial maps $\mathbb{A}^n \to \mathbb{A}^m$. Morphisms $\phi : X \to Y$ are in one to one correspondence with algebra homomorphisms on the respective coordinate rings (loosely speaking, the "pullback map") $\phi^\# : K[Y] \to K[X]$, where we work over some ground field K. For further discussion, see e.g., [106]. In "practice", it will be enough for our purposes to specify such a morphism by indicating how the m-tuple of polynomials $\phi = (\phi^1, ..., \phi^m)$ restricts onto Y, where the generalization to a rational morphism $\phi : X \dashrightarrow Y$ just involves taking ratios of polynomials. For the most part we will often discuss the "simplest" yet still non-trivial case of rational morphisms between affine spaces.

will also need a notion of a Lagrangian density $\mathcal{L}[\phi]$, which, for a fixed ϕ, is locally just a polynomial over a finite field. The action are given by evaluating $\mathcal{L}[\phi]$ at all the points of X and summing up. Or alternatively, we can just evaluate S at all the points and take the product:

$$\prod_{x \in X} \exp\left(\frac{2\pi i}{p} S_x\right). \tag{4.3}$$

We will impose a notion of "unitarity" by which we mean that the complex phases all have norm one. We enforce this through the condition that the evaluation of the action in this way produces a quantity valued in $\mathbb{F}_p$. Finally, the "limits of integration" Φ_i and Φ_f indicate fixed values of our morphism at marked locations on X specified by the divisors $t_i = 0$ and $t_f = 0$ We return to the quantum interpretation in section 5.

The rest of this section is organized as follows. We begin by developing a notion of physics over the finite field $\mathbb{F}_p$. We follow this with a discussion of finite field extensions such as $\mathbb{F}_q$, and finally the algebraic closure $\overline{\mathbb{F}}_p$. We then extend this to varieties over finite fields in characteristic p.

4.1 Physics on $\mathbb{F}_p$

We begin by revisiting our discretized bosonic system, but now with an eye towards maintaining additional analytic structure. We do this so that we can keep additional symmetries manifest, and also so that we can eventually generalize to systems with other sorts of degrees of freedom (such as fermions, vector bosons and gravitons). We discuss an alternative way to build finite characteristic actions in Appendix H, as well as some of the difficulties in generalizing it to more general geometries.

Let us return, then, to nearly the beginning. We now posit that we are working with a quantum system with integer values for our fields and in which the reduced Planck constant is discretized in units of p:

$$\hbar = \frac{p}{2\pi}. \tag{4.4}$$

We also assume that all observables of interest are really specified modulo p. For example, we assume the kinetic and potential energies of the action take values in the integers, and that the physical operators of interest are all specified by fields modulo p. Some important examples to keep in mind include exponentiated fields which take values in the character group for $(\mathbb{F}_p, +)$, viewed as an additive group.

Working over the integers modulo p, we arrive at a finite field, $\mathbb{F}_p$.[6] We review some properties of finite fields in Appendix C, and we refer the interested reader there for a brief discussion of this rich subject. Compared with more familiar fields such as the rational numbers, real numbers or complex numbers (or even the p-adics), adding up any element

[6]Unfortunately, the terminology "field" will be used for different sorts of objects. Hopefully the context will be clear.

$\phi \in \mathbb{F}_p$ by a multiple of p results in zero, namely $p\phi = 0$. A field which satisfies this property is said to be in characteristic $p > 0$. If this property does not hold, we say that the field is in characteristic zero.

One consequence of this is that there is no natural notion of a metric we can provide, though as we explain, there is a close characteristic p analog which retains some of the structure one would want of a physical metric. That being said, many analytic structures of geometry do remain intact provided we are flexible in our notion of what counts as a physical morphism.

Having specified that our physical fields actually take values in a finite field, we could in principle just repeat our lattice construction now, by specifying for each point $x \in X_{\text{spacetime}}$ on the spacetime lattice a value $\phi(x) \in \mathbb{F}_p$. So, we can view $X_{\text{spacetime}} \simeq \mathbb{Z}^D$ as a D-dimensional lattice with one direction singled out for time. We will shortly generalize this to move away from this limited choice.

As we have already mentioned, the notion of a finite derivative is a bit awkward, especially when there is no natural notion of "metric." To develop a suitable replacement, we will first consider a natural class of objects given by polynomials in some number of variables, written as $\mathbb{F}_p[u_1, ..., u_D]$. Our physical field ϕ can now be viewed as a polynomial in these variables:

$$\phi(u_1, ..., u_D) = \sum_{i_1,...,i_D} \phi_{i_1...i_D} (u_1)^{i_1} ... (u_D)^{i_D}, \tag{4.5}$$

where each of the $\phi_{i_1 \cdots i_D}$ is an element of $\mathbb{F}_p$. Taking a derivative proceeds just as in ordinary calculus. Note that when the exponent is a multiple of p, this derivative is automatically zero, a consequence of working in characteristic p. In principle, one can just continue to take ordinary derivatives, but a slightly more sophisticated option is to consider a Hasse derivative.[7] With all of these considerations in mind, we see that rather than dealing with a finite difference, it is in some sense simpler to work with derivatives of polynomials. Of course, once we evaluate the derivative we just compute the polynomial at the prescribed (integral) spacetime point, reduced modulo p. As an additional comment, we note that the space of polynomials is of course infinite. To generate concrete approximations we can always truncate to a fixed degree. This can then be used to match up with the lattice approximation.

Defining actions of relevance for physical systems is now straightforward. We illustrate by way of example. Given a polynomial $\phi \in \mathbb{F}_p[u_1, ..., u_D]$, we introduce a Lagrangian density $\mathcal{L}[\phi]$ as a functional on a given choice of ϕ. By composition of maps, this can also specifies

[7] In a variable u, the rth Hasse derivative is defined via its action on a monomial u^n: $\mathcal{D}^{(r)} u^n = \frac{n!}{n!(n-r)!} u^{n-r}$ when $0 \leq r \leq n$, and otherwise vanishes. The advantage of using this derivative is that it allows more terms to remain non-zero. Another helpful feature is that an analog of Taylor's theorem holds in terms of a local parameter u: $f = \sum\limits_{r} \mathcal{D}^{(r)}(f) \cdot u^r$.

an element of $\mathbb{F}_p[u_1, ..., u_D]$. As a specific example, we take:

$$\mathcal{L}[\phi] = \alpha \left((\partial_1 \phi)^2 - (\partial_2 \phi)^2 - ... - (\partial_D \phi)^2 \right) - V(\phi), \tag{4.6}$$

where $\alpha \in \mathbb{F}_p$ and $V \in \mathbb{F}_p[\phi]$. To extract a number, we now sum over the points in the spacetime using the evaluation map:

$$S[\phi] = \sum_{(x_1, \cdots, x_D) \in X_{\text{spacetime}}} \mathcal{L}(u_1 = x_1, ..., u_D = x_D), \tag{4.7}$$

where now we simply treat $\mathcal{L}$ as a polynomial in the formal parameters of $\mathbb{F}_p[u_1, ..., u_D]$, and then evaluate. In this case, the phase factor of the path integral defines a character map on the additive group of the finite field:

$$\exp : (\mathbb{F}_p, +) \to U(1) \tag{4.8}$$

$$S \mapsto \exp\left(\frac{2\pi i}{p} S\right). \tag{4.9}$$

As it stands, we are summing over a lattice with an infinite point set. This means in particular that S evaluated on this physical field configuration may not be well-behaved. On the other hand, since $\phi^p = \phi$ in $\mathbb{F}_p$, we are typically summing over "multiple copies" of the same spacetime point when we evaluate over all the integers.

One possibility is to just reduce the lattice $\mathbb{Z}^D$ modulo p, so that we instead deal with D-dimensional affine space in characteristic p:[8]

$$X_{\text{spacetime}} = \underbrace{\mathbb{A}^1(\mathbb{F}_p) \times ... \times \mathbb{A}^1(\mathbb{F}_p)}_{D \text{ times}} = \mathbb{A}^D(\mathbb{F}_p). \tag{4.10}$$

In the spirit of algebraic geometry, we can also consider more general algebraic varieties in characteristic p. These more general choices can have more or less points depending on the choices of hypersurface equations. See also Appendix D.

With this in mind, we can already anticipate that it will be fruitful to expand our horizons, allowing X to be specified as the zero set of more general polynomials in characteristic p. By a similar token, we can also enlarge the target space Y in a similar way. In all these cases, there is a suitable generalization of a polynomial to maps of the form:

$$\phi : X_{\text{spacetime}} \to Y_{\text{target}}, \tag{4.11}$$

where so far, we have restricted to affine spaces. The main idea is to view these ϕ's as locally specified by polynomials, and to then construct a Lagrangian from these fields.

A surprising feature of our Lagrangian is that the kinetic term of our scalar field theory

[8] Recall that $\mathbb{A}^D(\mathbb{F}_p) = \text{Spec}\,(\mathbb{F}_p[u_1, ..., u_D])$, and that as a point set, the affine line $\mathbb{A}^1(\mathbb{F}_p)$ is just $\mathbb{F}_p$.

seems to make reference to a Lorentzian signature metric. Of course, this is an illusion; in characteristic p we also have:

$$(\partial_1\phi)^2 - (\partial_2\phi)^2 - ... - (\partial_D\phi)^2 = (\partial_1\phi)^2 + (p-1)((\partial_2\phi)^2 + ... + (\partial_D\phi)^2), \tag{4.12}$$

so if we naively lift back to characteristic zero, we could view our Lorentzian signature "metric" as actually specifying a Euclidean signature metric, but with different weighting for the spatial and Euclidean time directions. That being said, the conjugacy class of the different quadratic forms are indeed different, even in characteristic p, so there is still a meaningful distinction captured by the signature of a quadratic form.

4.2 Physics on $\mathbb{F}_q$

Our discussion thus far has focused on varieties defined over $\mathbb{F}_p$, the integers modulo p. We now extend these considerations to other finite fields such as $\mathbb{F}_q$. Recall from Appendix C that every finite field in characteristic p has $q = p^n$ elements for some $n \geq 1$. The field $\mathbb{F}_q$ can be constructed as the splitting field of an irreducible degree n polynomial over $\mathbb{F}_p$. We can think of this field as obtained by adjoining a single root α of such a polynomial, writing $\mathbb{F}_q = \mathbb{F}_p(\alpha)$. From the perspective of Galois theory, we can view $\mathbb{F}_q$ as a vector space over the field $\mathbb{F}_p$. A convenient basis of vectors is given by the p^{th} powers of this root, so we can represent any element in $\mathbb{F}_q$ as a power series of the form:

$$\phi = \sum_{j=1}^{n} \phi_j \alpha^{p^{j-1}}, \tag{4.13}$$

for $\phi_j \in \mathbb{F}_p$. Indeed, the Frobenius automorphism:

$$F : \mathbb{F}_q \to \mathbb{F}_q \tag{4.14}$$

$$\phi \mapsto \phi^p \tag{4.15}$$

simply pemutes these powers. Here, we have used the fact that in characteristic p, $(a+b)^p = a^p + b^p$.

The appearance of an n-dimensional vector space over $\mathbb{F}_p$ has a clear interpretation in terms of the physical degrees of freedom we have already introduced. Instead of considering a single physical field moving in a 1D target spanned by the integers, we can consider n such physical fields. We can denote this by an n-component vector with coordinates $\phi^{(1)}, ..., \phi^{(n)}$. This also makes it clear that we can construct corresponding Lagrangians involving our n physical fields. As an example, we can construct a kinetic term for our n physical fields given by:

$$G_{AB} \left(\partial_1\phi^A \partial_1\phi^B - \partial_2\phi^A \partial_2\phi^B - ... - \partial_D\phi^A \partial_D\phi^B \right), \tag{4.16}$$

where repeated indices are summed over. Here, we have introduced a symmetric bilinear form with entries G_{AB}:

$$G : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p, \tag{4.17}$$

which specifies a "dot product" for the system.

Now, instead of working in terms of these n-component vectors, we could alternatively view this as a single physical field on a one-dimensional target $\mathbb{F}_q$. In characteristic zero, we implicitly do exactly this sort of thing when we view a complex scalar field as a linear combination of two real scalars. In contrast to the complex numbers, however, there are many analogs of the imaginary numbers which we can adjoin to $\mathbb{F}_p$. As explained in Appendix C, we can alternatively view the bilinear forms of line (4.17) as an $\mathbb{F}_p$ valued pairing:

$$G : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p. \tag{4.18}$$

So, it is a matter of taste whether we wish to work in terms of many physical fields, or in terms of a single $\mathbb{F}_q$ valued field. The main condition we need to enforce in this generalized perspective is that our action takes values in $\mathbb{F}_p$ rather than the larger field $\mathbb{F}_q$. Arguing by analogy with other quantum systems, we need to ensure that there is a proper notion of "unitary time evolution," and this would be destroyed if our action ended up being valued outside the ground field, i.e., the integers modulo p.[9]

One way to construct $\mathbb{F}_p$ valued actions is to demand that all evaluations are invariant under the Frobenius automorphism F with $F(\phi) = \phi^p$. Indeed, $\mathbb{F}_p$ is the only subfield of $\mathbb{F}_q$ invariant under this automorphism. Given an element $\phi \in \mathbb{F}_q$, common invariants include the Trace and Norm:

$$\mathrm{Trace}(\phi) = \sum_{i=0}^{n-1} F^i(\phi) \tag{4.19}$$

$$\mathrm{Norm}(\phi) = \prod_{i=0}^{n-1} F^i(\phi). \tag{4.20}$$

The Trace is clearly useful in producing invariant kinetic terms, while both the Trace and Norm are useful in constructing invariant potential energy densities. At a more general level, our only true demand is that our action have "local" interaction terms. There are, however, physically motivated choices, as we have indicated above.

Having seen that we can extend the target space to be $\mathbb{F}_q$, one might ask whether a similar extension to spacetimes defined over $\mathbb{F}_q$ is well-motivated. Of course, at a formal level, nothing stops us from doing so. Indeed, so long as our action continues to evaluate to elements in $\mathbb{F}_p$, there is no reason not to make this extension. From a physical perspective,

[9]Of course, the notion of "unitarity" is less clear in this setting because even our notion of "time ordering" in the characteristic p setting is somewhat less clear. There is, however, still a notion of past, present and future as dictated by the degree of terms in local Laurent expansions. We provide further discussion on such Hilbert space considerations in section 5. We thank S. Cecotti for several insightful criticisms on this point.

one can view this as supplementing our original discretized spacetime by additional points. A perhaps more satisfying answer is that the target space for a string is interpreted as another spacetime in its own right. In this context, then, it is again sensible to allow for such field profiles.

4.3 Physics on $\overline{\mathbb{F}}_p$

Proceeding in this way, we can now ask about the interpretation of taking the target space to be $\overline{\mathbb{F}}_p$, the algebraic closure of our finite field. One might view this as playing the analogous role to that which the complex numbers play in relation to the real number numbers. Of course, here, there are many more analogs of the "imaginary numbers" available!

One important remark is that the algebraic closure has infinite order. This means that the procedure for computing values of the action used previously will not really work, since the "evaluation map" procedure requires us to sum over all the points of a variety. Now, in characteristic zero we could introduce a measure on our spacetime and use this to suitably integrate over the Lagrangian density.

What can we do in the present case? The main idea we use to define the path integral phase factor $\exp(iS/\hbar)$ in this case is to observe that actually, our evaluation can instead be viewed as a product over characters. Recall that for a field K, the group of additive characters involves an "exponential map" to $U(1) \subset \mathbb{C}^{\times}$. Phrased in this way, we can, for each point in a finite field first compute the additive character, and only then take the product. In the obvious notation, the evaluation of the phase factor for the path integral can instead be written as:

$$\prod_{x \in X} \exp\left(\frac{2\pi i}{p} S_x\right) \in U(1) \subset \mathbb{C}^{\times}. \tag{4.21}$$

The advantage of setting things up this way is that now, we can consider a sequence of containments:

$$K_0 \subset K_1 \subset K_2 \subset ... \subset K_m \subset ... \subset \overline{\mathbb{F}}_p, \tag{4.22}$$

and with it the corresponding sequence of characters obtained from evaluation on a given field configuration:

$$\chi_{K_0}, \chi_{K_1}, ..., \chi_{K_m}, \cdots \tag{4.23}$$

Of course, there is no guarantee that such a sequence will converge in the metric topology of $\mathbb{C}$. Additionally, there is of course more than one way to build a nested containment of finite field extensions contained in $\overline{\mathbb{F}}_p$. To have a well-defined limit, we require that any such sequence converges to the same point in $U(1)$, and when it does, we write the limit as:

$$\varinjlim \chi_{K_n} \equiv \chi_{\overline{\mathbb{F}}_p}. \tag{4.24}$$

At a practical level, however, we can simply truncate a given sequence. Proceeding in this way, we can speak of path integrals over algebraically closed fields such as $\overline{\mathbb{F}}_p$.

In this enlarged setting we can also contemplate the physical meaning of the Frobenius automorphism, namely the generator of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. For finite fields $\mathbb{F}_q$ we interpreted the Frobenius map as permuting a collection of $\mathbb{F}_p$ valued fields, and the same considerations apply here as well, albeit for a now infinite collection of physical fields. This provides another way for us to interpret our path integral phase factor of line (4.21).

4.4 Physics on Varieties in Characteristic p

Our discussion so far has mainly focused on the simplest examples of spacetimes and target spaces. We can also consider more general geometries by specifying varieties in characteristic p. The procedure for constructing such spaces is a standard one from algebraic geometry, and it carries over essentially unchanged. We construct affine patches of a variety by specifying the zero set for some polynomials. Then, we glue these patches together to produce a our more general variety.[10] We now speak of our physical fields as specified by rational maps of the form:

$$\phi : X_{\text{spacetime}} \dashrightarrow Y_{\text{target}}. \tag{4.25}$$

In terms of the local coordinate rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ for $x \in X$ and $y \in Y$, this means that we will allow our physical fields y to be written as ratios:

$$\phi = \frac{P}{Q}. \tag{4.26}$$

The reason we should allow such maps is that in most geometries of interest, working with just polynomials will not produce enough "interesting" maps. This is the point of allowing birational maps. The price we pay in doing this is that we inevitably encounter possible singularities in the evaluation of our action. This is actually not that problematic, it just indicates the physical presence of a source, and means that we need to specify some choice of boundary conditions in the path integral with prescribed pole structure for field configurations. This is often referred to as inserting a defect operator in the path integral. This is also customary in specifying asymptotic scattering states. For all these reasons, we shall remain flexible in our notion of a physical field. The proper notion of the path integral would seem to involve summing over rational morphisms $X_{\text{spacetime}} \dashrightarrow Y_{\text{target}}$.

In this more general setting, we can now also provide a more geometric formulation for the terms appearing in our action. Consider, for example, the kinetic term of a bosonic field theory. In characteristic p, we can still speak of the cotangent space to a point, so we can

[10]The extension to schemes and stacks poses no additional complications other than having to carry around more formalism. We therefore leave it implicit in what follows.

consider the pullback map on the cotangent spaces:

$$\phi^* : T_y^* Y \dashrightarrow T_x^* X, \tag{4.27}$$

a differential such as $d\phi$ has the standard local form:

$$d\phi^A = \frac{\partial \phi^A}{\partial x^a} dx^a. \tag{4.28}$$

Indeed, as we have repeatedly emphasized, much of the algebro-geometric structure typically used in characteristic zero carries over to characteristic p (with suitable amendments).

Additionally, we can introduce symmetric bilinear forms:[11]

$$G : T^*Y \otimes T^*Y \to \mathbb{F}_q \to \mathbb{F}_p \tag{4.29}$$
$$h : T^*X \otimes T^*X \to \mathbb{F}_q \to \mathbb{F}_p, \tag{4.30}$$

where we have factored this map through the Trace map. We refer the reader to Appendix D for the definition of the cotangent space in characteristic p. The main point is that even though there is little notion of "distance," in these spaces, we can still introduce symmetric bilinear forms valued on the "observable" numbers.

Consequently, we can now specify far more general actions in characteristic p as well. Superficially, there is little change from our earlier considerations. For example, a non-linear sigma model metric on a spacetime X can be written as:

$$S = \sum_{x \in X} \sqrt{\det h}\, h^{ab} G_{AB} \partial_a \phi^A \partial_b \phi^B. \tag{4.31}$$

Here, each of the ϕ's is to be interpreted as a rational map from $X \dashrightarrow Y$, and derivatives of local coordinates on X are computed as before. Again, our only demand at this point is that for each such physical field configuration, the action remains valued in $\mathbb{F}_p$, this being the analog of unitarity in characteristic p. The appearance of "$\sqrt{\det h}$" is really a stand-in for the scalar dual to the "volume-form" in $\Omega^m(X, \mathcal{K}_X)$. This volume-form implicitly depends on the bilinear h_{ab}. See subsection 4.5 for further discussion.

Observe also that the expression we have arrived at is naturally covariant, even though we are working on a discretized spacetime and target space. Indeed, under a non-singular (up to a lower codimension space) change of coordinates, the standard rules of tensor calculus hold. In characteristic p, the analog of a local analytic isomorphism (i.e., a diffeomorphism) is an étale morphism (a special case of a smooth morphism in which the relative dimension is zero). The main thing we want to ensure is that we have the characteristic p analog of the inverse function theorem for manifolds in characteristic zero. Demanding that the

[11]Here we use the physicist convention for a metric, $ds^2 = G_{AB} dY^A dY^B$ so we evaluate on the cotangent bundle rather than the tangent bundle.

Jacobian is invertible ensures this. See Appendix D for an extremely brief discussion of étale morphisms.

As a final generalization, now that we have moved to a far more geometric language, it is natural to ask whether we can start to incorporate some additional sorts of degrees of freedom, such as vector bosons and even gravitons. A priori, there does not appear to be any issue with doing this in characteristic p.

In fact, some of the mathematical formulation of vector bosons and gravitons in characteristic p has been carried out in reference [93]. The key feature for us is that using a suitable notion of localization of sheaves, the resulting formulae for the gauge connection and "metric" behave completely analogously to what one has in the characteristic zero case!

For example, for an abelian gauge field V_a, we can consider gauge transformations such as:

$$V_a \mapsto V_a + \partial_a \varepsilon, \tag{4.32}$$

where ε is to be interpreted locally as a polynomial in the coordinate ring of the variety. Note that the field strength:

$$F_{ab} = \partial_a V_b - \partial_b V_a \tag{4.33}$$

is invariant under such gauge transformations, independent of the characteristic. Quantities such as $F_{ab}F^{ab}$ can then be used to build gauge invariant actions in the standard way. Here, we raised and lowered indices with a symmetric bilinear form h^{ab}.

Constructing a scalar degree of freedom charged under such a field is also straightforward. For example, given α, we can impose the condition for gauge transformations:

$$\alpha \mapsto \alpha - \varepsilon, \tag{4.34}$$

so the quantity:

$$\frac{1}{2} g^{ab}(\partial_a \alpha + V_a)(\partial_b \alpha + V_b) \tag{4.35}$$

is also gauge invariant.

We now provide a more systematic treatment of gauge interactions, but still focused primarily on motivated examples. From the outset, one complication we face is that in characteristic zero we can easily pass from elements of a Lie algebra to a local presentation of an element in the Lie group via the exponential map. In characteristic p more caution is warranted but the general formalism of Lie groups and their relation to Lie algebras can still be formulated, as in reference [107], and we refer the interested reader there for additional details.

We begin by constructing a physical field theory with $SO(n, \mathbb{F}_p)$ gauge interactions.[12]

[12]Implicit in our discussion of this group is a choice of quadratic form on an n-dimensional vector space. In fact, for p an odd prime, when n is odd, there is a single congruence class so there is no ambiguity, while when n is even, there are two distinct choices which are often labelled as $SO^{\pm}(n, \mathbb{F}_p)$, referring to the choice

Consider the theory of an n-component vector of $\mathbb{F}_p$ valued scalar fields which we denote as ϕ^A. We introduce a fixed symmetric bilinear form $G_{AB} = \delta_{AB}$. We can then consider Lagrangians such as:

$$L = \frac{1}{2} G_{AB} \partial_a \phi^A \partial^a \phi^B - \lambda \left(G_{AB} \phi^A \phi^B - \xi \right)^2 , \tag{4.36}$$

with λ and ξ fixed parameters. We observe that this Lagrangian enjoys an $SO(n, \mathbb{F}_p)$ symmetry. By this, we mean the set of $n \times n$ matrices with entries in $\mathbb{F}_p$ such that $M^T M = \mathbb{I}_{n \times n}$, the identity.

We now attempt to gauge this global symmetry. In characteristic zero, we would introduce local gauge transformations, as designated by g_x, so that for each point $x \in X_{\text{spacetime}}$, we get an element in the symmetry group. We would like to attempt something similar in characteristic p. The first complication we encounter is that all our fields are being represented as polynomials, so one might rightly ask whether this can be extended to the present setting. Indeed, the proper framework for carrying this out is to consider a sheaf $\mathcal{V}$ such that each stalk $\mathcal{V}_x$ admits a group action by $SO(n, \mathbb{F}_p)$. Then, we can speak of the condition $g_x^T g_x = \mathbb{I}_{n \times n}$. To get this into a more practical form recognizable to a physicist, we can also consider the space of $n \times n$ matrices with entries in $\mathbb{F}_p(t)$, the field obtained by adjoining the formal element t. Then, the condition $g(u)^T g(u) = \mathbb{I}_{n \times n}$ specifies a set of $n \times n$ matrices with entries in $\mathbb{F}_p(t)$ which satisfy the desired gauge transformation properties. With this in place, we can now introduce a vector potential V_a. Near a point $x \in X$, each component of this vector is to be viewed as an element of $\mathfrak{so}(n, \mathbb{F}_p) \otimes \mathcal{O}_{X,x}$, namely we impose the condition $(V_a)^T = -V_a$ on our local polynomial expressions. Globally, of course, we should think of $\partial_a + V_a$ as specifying a connection on our sheaf. We wish to consider gauge transformations of the form:

$$V_a \mapsto g^{-1} V_a g + g^{-1} \partial_a g, \tag{4.37}$$

where each g is interpreted as above. The important point for us is that even though we do not have the exponential map, we can still consider the group action of $SO(n, \mathbb{F}_p)$ on $\mathfrak{so}(n, \mathbb{F}_p)$.

At this point, the discussion is so close to that of characteristic zero that we can simply write down the standard action obtained from minimal coupling:

$$L = \frac{1}{2} G_{AB} (\partial_a \phi^A + (V_a)^A{}_{A'} \phi^{A'})(\partial^a \phi^B + (V^a)^B{}_{B'} \phi^{B'}) - \lambda \left(G_{AB} \phi^A \phi^B - \xi \right)^2 . \tag{4.38}$$

We can also extend this to other characteristic p fields such as $\mathbb{F}_q$, as per our discussion in earlier sections.

A pleasant feature of the group $SO(n, \mathbb{F}_p)$ is that all entries are already valued in $\mathbb{F}_p$, so the characteristic p analog of "unitarity" is guaranteed. What about other gauge groups?

of signature of the quadratic form. When $p = 2$ similar considerations hold but one must instead reference the Arf invariant of the quadratic form. We thank S. Cecotti for helpful comments on this point.

Perhaps the most familiar in physics applications characteristic is the group $U(n)$. To get something like that in the present context, we need to have a suitable notion of hermitian conjugation, as well as a suitable notion of an "imaginary number."

The appropriate notion of complex conjugation in characteristic p is Frobenius conjugation. Working over a ground field $\mathbb{F}_q$, there is a notion of Frobenius conjugation given by $F_q(\phi) = \phi^q$ which holds fixed all elements of $\mathbb{F}_q$. We can then consider a quadratic extension by an element $\widehat{i}_q$ specified by the condition:

$$F_q(\widehat{i}_q) = -\widehat{i}_q, \tag{4.39}$$

which acts as the characteristic p analog of complex conjugation. We remark that this element may not square to -1. For example, in $\mathbb{F}_5$, observe that $3^2 = -1$ but that $3 \in \mathbb{F}_5$ whereas $\widehat{i}_5$ is not (since it is not invariant under Frobenius conjugation). To proceed more systematically, we will instead seek out a root of the polynomial equation:

$$x^q = -x, \tag{4.40}$$

and we denote one such root by $\widehat{i}_q$. Observe that by design, we have:

$$F(\widehat{i}_q) = (\widehat{i}_q)^q = -\widehat{i}_q. \tag{4.41}$$

Since $\widehat{i}_q$ is not invariant under Frobenius conjugation, it is not an element of $\mathbb{F}_q$. Note, however, that its square $(\widehat{i}_q)^2$ is invariant, and is therefore an element of $\mathbb{F}_q$.

We can now introduce an analog of hermitian conjugation as follows. Given an $n \times n$ matrix M with entries in $\mathbb{F}_q(\widehat{i}_q)$, write:

$$H = H_1 + \widehat{i}_q H_2, \tag{4.42}$$

with H_1 and H_2 some $n \times n$ matrices with entries in $\mathbb{F}_q$. We define a daggering operation:

$$H^\dagger \equiv H_1^T - \widehat{i}_q H_2^T. \tag{4.43}$$

The group of unitary matrices is now defined by writing:

$$U(n, \mathbb{F}_q(\widehat{i}_q)) = \left\{ H \in GL\left(n, \mathbb{F}_q(\widehat{i}_q)\right) | H^\dagger H = \mathbb{I}_{n\times n} \right\}. \tag{4.44}$$

Now, in characteristic zero, we could start with the theory that enjoys an $SO(2n)$ global symmetry and consider gauging a subgroup such as $U(n)$. In characteristic p, this is a bit more subtle because the choice of hermitian conjugation now makes reference to a specific choice of $\widehat{i}_q$.[13]

[13]In some cases, however, it is possible to proceed in a similar fashion to characteristic zero. To illustrate, suppose that we fix our ground field to be $\mathbb{F}_p$. We then make the further assumption that $\widehat{i}_p^2 = -1$. Note

We now build a Lagrangian which enjoys the global symmetry $U(n, \mathbb{F}_p(\widehat{i}_p))$. Consider a theory of $2n$ $\mathbb{F}_p$-valued scalars $\phi^1, ..., \phi^{2n}$. We construct the "complexified" combinations:

$$\varphi^A = \phi^A + \widehat{i}_q \phi^{A+n} \tag{4.47}$$

$$\overline{\varphi}^{\overline{A}} = \phi^A - \widehat{i}_q \phi^{A+n}. \tag{4.48}$$

In this case, we can introduce a suitable bilinear pairing $G_{A\overline{B}}$ and write:

$$L = G_{A\overline{B}}(\partial_a \varphi^A + (V_a)^A{}_{A'}\, \varphi^{A'})(\partial^a \overline{\varphi}^{\overline{B}} - \left(\overline{V}^a\right)^{\overline{B}}{}_{\overline{B}'}\, \overline{\varphi}^{\overline{B}'}) - \lambda \left(G_{A\overline{B}} \varphi^A \overline{\varphi}^{\overline{B}} - \xi\right)^2. \tag{4.49}$$

Here, we have introduced a vector potential V_a. Near a point $x \in X$, each component of V_a is to be viewed as an element of $\mathfrak{u}(n, \mathbb{F}_p(\widehat{i}_p)) \otimes \mathcal{O}_{X,x}$, namely we impose the condition $(V_a)^\dagger = -V_a$ on our local polynomial expressions. Globally, of course, we should think of $\partial_a + V_a$ as specifying a connection on our sheaf.

Turning next to the analog of the graviton, we have also mentioned that there is really no issue in defining a symmetric bilinear form defined over each point of a scheme X. We mainly need to impose a local equivalence relation:

$$h_{ab} \sim h_{ab} + \partial_a \nu_b + \partial_b \nu_a, \tag{4.50}$$

for $\nu \in T^* X$. Summing over all equivalence classes in this way provides the path integral instruction for how to sum over the space of such "metrics."

As a final amusing comment, note that the definition of standard Riemannian geometry tensors only makes algebraic reference to the quantity h_{ab} and its derivatives. This would seem to suggest that we can even borrow the standard actions for gravity, including higher derivative interactions.

4.5 Further "Metric" Considerations

In the previous subsection we briefly mentioned the action for a non-linear sigma model, which makes reference to a choice of symmetric bilinear form on both the spacetime and

that this restricts us to $p = 3 \bmod 4$. To see why, write $p = 3 + 4n$. Next, observe that

$$-\widehat{i}_p = \left(\widehat{i}_p\right)^p = \left(\widehat{i}_p\right)^{3+4n} = \left(\widehat{i}_p\right)^3, \tag{4.45}$$

which implies:

$$-1 = \left(\widehat{i}_p\right)^2. \tag{4.46}$$

In this case, we can use the standard manipulations used in characteristic zero. This, however, imposes restrictions on the prime p, a feature which we would like to avoid.

target space. Recall that we took a general action of the form:

$$S = \sum_{x \in X} \sqrt{\det h} h^{ab} G_{AB} \partial_a \phi^A \partial_b \phi^B. \tag{4.51}$$

where we have reproduced equation 4.31 for the convenience of the reader. In the above, we have made reference to a "metric" on X, by which we mean a symmetric bilinear form $T^*X \otimes T^*X \to \mathbb{F}_p$. Now, taking the determinant of this bilinear form can be presented locally in terms of a polynomial in the local coordinates. The somewhat unpleasant feature of this presentation is that there is no guarantee that $\det h$ is actually a polynomial. Formally speaking, nothing stops us from including such quantities. For example, we can simply take the ring of functions and adjoin various finite extensions. On the other hand, doing so runs a bit counter to the philosophy of only allowing quantities which have a suitable analytic presentation.

To explain why this is still natural, even in the characteristic p setting, it is helpful to return to the characteristic zero setting for X a Riemannian geometry. Now, in characteristic zero, there is a clear justification for including the pre-factor $\sqrt{\det h}$. This is because, in terms of local coordinates on a manifold X, the volume form can be presented in terms of this metric data:

$$\mathrm{dVol}_X = d^m x \sqrt{\det h}. \tag{4.52}$$

Indeed, under a local coordinate transformation $x^a \mapsto f^a(x)$, the transformation of $dx^1 \wedge ... \wedge dx^m$ is precisely encoded in the corresponding Jacobian of $\partial f^a / \partial x^b$.

What is the justification for including such a pre-factor in the characteristic p setting? On the one hand, we are summing over a finite number of points, and an automorphism of X will still produce the same collection of points. Moreover, the kinetic term $h^{ab}\partial_a \phi \partial_b \phi$ is already invariant under coordinate reparameterizations. Said differently, the Lagrangian densities we have been discussing can be viewed as zero-forms in the local ring of functions, namely, elements of $\Omega^0_X(\mathcal{O}_X)$. At the level of homology groups, we also have the statement of Serre duality over the ground field K [108]:

$$H^i(X, \mathcal{E}) \times H^{m-i}(X, \mathcal{E}^\vee \otimes \mathcal{K}_X) \to K, \tag{4.53}$$

so upon taking $i = 0$ and $\mathcal{E} = \mathcal{O}_X$, we see there is a natural notion of "integration" which will indeed return an element of K. In the case of $K = \mathbb{C}$ and X a Calabi-Yau space, the corresponding section of $H^m(X, \mathcal{K}_X)$ is just the volume form $\overline{\Omega} \wedge \Omega$, with Ω a holomorphic $(m, 0)$-form, which is uniquely specified (in the Calabi-Yau case) up to an overall non-zero complex number.

More generally, then, we can still speak of introducing a volume form as given by a section of $H^m(X, \mathcal{K}_X)$, and this will canonically pair with the zero-form, as specified by a Lagrangian. The choice of a particular section is our stand-in for "$\sqrt{\det h}$", and so in this sense it is indeed appropriate to include this explicit factor, even in the characteristic p

setting. Note that this also automatically answers another potentially troublesome point about possibly taking the square-root of a local function in the characteristic p setting. Rather than artificially introducing a notion of square-root, we are instead specifying a particular choice of "volume-form", viewed as an element of $\Omega^m(X, \mathcal{K}_X)$. Varying the bilinear $h : T^*X \otimes T^*X \to K$ implicitly changes the element of $\Omega^m(X, \mathcal{K}_X)$ we use, but all of this is again specified without recourse to a particular field extension. The meaning of $\sqrt{\det h}$ is then the scalar in $\Omega^0(X, \mathcal{O}_X)$ dual to this volume-form.

A further comment here is that in some cases, the "volume" of X specified by h can end up being zero. This is really just part of the price we pay for working in characteristic p. For example, on the affine line over $\mathbb{F}_q$, summing over all points with respect to the "flat metric" where $\det h = 1$ yields:

$$\sum_{x \in \mathbb{A}^1(\mathbb{F}_q)} 1 = p^n = 0, \tag{4.54}$$

with $q = p^n$. On the other hand, on the punctured affine line where we remove the origin, we instead get:

$$\sum_{x \in \mathbb{A}^\times(\mathbb{F}_q)} 1 = p^n - 1 = -1. \tag{4.55}$$

Choosing a different value for $\sqrt{\det h}$ results in a different "volume". For example, on the affine line over $\mathbb{F}_q$, we could instead take $\sqrt{\det h} = x^{q-1}$. In this case, we have $x^{q-1} = 1$ for $x \neq 0$, and so we get:

$$\sum_{x \in \mathbb{A}^1(\mathbb{F}_q)} x^{q-1} = \sum_{x \in \mathbb{A}^\times(\mathbb{F}_q)} x^{q-1} = p^n - 1 = -1, \tag{4.56}$$

namely we get a non-zero answer. Note that in this case, $\sqrt{\det h}$ vanishes at $x = 0$.

4.6 Statistical Formulation

Our emphasis here has been on formulating a notion of a path integral in which we explicitly reference the usual complex phase factors which appear in quantum systems. Now, a common approach to the study of path integrals in characteristic zero is to consider the analytic continuation of the spacetime to a Euclidean signature theory, in which case we really have a statistical field theory with a Boltzmann factor $\exp(-S_E)$. One reason to do this is that in the Euclidean setting, statistical field theories have potentially better convergence properties (though notably gravity is an exception to this general claim).

So, it is reasonable to ask whether we can set up a Euclidean signature version of our characteristic p action. At the level of specifying the "metric" and its Euclidean signature analogs, there is no apparent issue in specifying a Euclidean signature action. In fact, it is fruitful to consider the family of actions $S[h]$ as a function of the symmetric bilinear form h which has support on $T^*X \otimes T^*X$. Then, we can view one choice of h as specifying our

Lorentzian signature bilinear form, and another as specifying a Euclidean signature bilinear form. So, at least at the level of specifying what we mean by an action in Euclidean signature, there is no issue.[14]

The complications arise in specifying what we could mean by the Boltzmann factor $\exp(-S_E)$ in this setting.[15] Indeed, our main setup has exploited the characters of the additive group $(\mathbb{F}_p, +)$, and this requires that we evaluate quantities via a character χ_S such as:

$$\chi_S = \exp\left(\frac{2\pi i}{p} S\right), \tag{4.57}$$

with S the action evaluated on a particular field configuration. If we attempt to analytically continue, we face the unpleasant feature that we are no longer constructing a well-defined character.

A workaround is available, however, because χ_S takes values in $\mathbb{C}^\times$. Consequently, we can, up to a choice of branch cut, consider:

$$\frac{p}{2\pi i} \log \chi_S = S + m \tag{4.58}$$

for some integer $m \in \mathbb{Z}$. Making a non-canonical choice of branch cut, we can set $m = 0$, and then a Boltzmann factor can be specified for the Euclidean signature action as:

$$\exp(-S_E) \equiv \exp\left(-\frac{p}{2\pi i} \log \chi_{S_E}\right). \tag{4.59}$$

Performing a path integral then involves summing over our space of morphisms.

A more formal way to state the same prescription is to first consider a primitive p^{th} root of unity:

$$\zeta \equiv \exp(2\pi i/p). \tag{4.60}$$

Then, we observe that because $\zeta^p = 1$, we expect any correlation function to be expressed as a rational function in the ζ. We can write this as:

$$\text{Correlator} = (\zeta^p)^m \frac{\sum\limits_{j=0}^{p-1} a_j \zeta^j}{\sum\limits_{j=0}^{p-1} b_j \zeta^j}, \tag{4.61}$$

for some coefficients $a_j, b_j \in \mathbb{R}$. While it is tempting to restrict these coefficients to lie in the integers, we should remain flexible, especially since we may need to regularize the infinite

[14]Another way to proceed is to view the Lagrangian density as a difference of kinetic and potential energy densities, namely $L = T - V$. Then, we can can consider the Euclidean signature Lagrangian as $L_E = T + V$, with the usual caveats about what we mean by time derivatives in the two settings.

[15]For various topological actions such as the Chern-Simons functional, the "Euclidean signature" version still has a complex phase factor, but that case is too specialized for what we are discussing here.

sums which appear in summing over all possible morphisms. Here, we have also allowed for an overall factor of $(\zeta^p)^m$, although of course this is just unity when we make use of equation (4.60). The previously discussed prescription of setting $m = 0$ to arrive at equation (4.59) can be formalized by treating $z = \zeta$ as valued in $\mathbb{C}$. Then, "all" we are doing is restricting to the case where z is again restricting to real values of $z = \exp(-2\pi/p)$ instead. The different choices of m just amount to an overall prefactor which is common to all correlators. As such, we can work with a canonical choice by absorbing this into our definition of the partition function.

Although rather natural at a formal level of manipulation, the process of analytically continuing an entire function based on how it behaves over a finite set of values (namely the primitive p^{th} roots of unity) is of course somewhat suspicious. Indeed, it has by now been appreciated in many places that even in characteristic zero, the process of analytic continuation in the signature can introduce various subtleties in the path integral prescription, see e.g., [109, 110] for recent discussion. At present, it would seem the best we could hope for is that the prescription outlined above is the one appropriate for capturing a leading order saddle point approximation, while contributions from subleading saddle points might be intrinsically ambiguous. In principle the ambiguity is completely resolved because we are dealing with a finite method for regulator quantum field theory correlators (much as in lattice gauge theory), but the precise way in which this shows up in the Euclidean signature formulation remains to be worked out.

It would be interesting to further investigate the convergence properties of this Euclidean signature action, as well as the implicit dependence on a choice of analytic continuation alluded to above.

5 Hilbert Space Considerations

Up to now, our main emphasis has been on developing a path integral formalism for physical theories in characteristic p. To a certain extent, this provides an operational definition of our physical theory, because we can use this framework to compute correlation functions of operators. Of course, what this leaves implicit is the actual structure of the Hilbert space in question. Our aim in this subsection will be to explain how this additional physical structure comes about in characteristic p.

To begin, we need to assume a notion of time in our characteristic p spacetime. For ease of exposition, we will assume that our spacetime X factorizes as $X_{\text{time}} \times X_{\text{space}}$, where we view X_{time} as the affine line or its projectivization, and X_{space} as the spatial directions. This can be generalized to fibrations of the form:

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ & & \big\downarrow \\ & & X_t \end{array}, \tag{5.1}$$

in the obvious notation. In what follows we leave this further generalization implicit. To emphasize this structure, we adopt the physics notation $(t, u_s) = u^a$ for local "spacetime" coordinates of X.[16] We can then still speak of rational morphisms such as:

$$\phi : X \dashrightarrow Y \tag{5.2}$$

to some target space Y.

We now argue that there is a local notion of past, present and future as defined by the local mode expansions in the X_t coordinate of our characteristic p spacetime.[17] This will again put the finger on the need for an infinite dimensional Hilbert space.

The main point is already conveyed in the one-dimensional setting where we have a single scalar field on the affine line with the origin deleted:

$$\phi : \mathbb{A}^{\times} \to \mathbb{A}, \tag{5.3}$$

over a fixed ground field $\mathbb{F}_q$. We view the origin as the far past, and specify the far future implicitly through the map $u \mapsto 1/F(u) = 1/u^p$, i.e., the inverse Frobenius conjugate. All of this makes sense on $\mathbb{P}^1(\mathbb{F}_q)$. In this way of thinking, we can view $\mathbb{A}^{\times}(\mathbb{F}_q)$ as defining a cylinder with $\mathbb{A}^{\times}(\mathbb{F}_p)$ specifying the time coordinate, with 0 in the far past and ∞ in the far future. See figure 2 for a depiction.

[16] In particular, in this section, t and u_s refer to the formal variables which will appear in various polynomial rings. We refer to the "evaluation points" on the spatial slice by x_s.

[17] "Nor is it right to say there are three times: past, present and future. Perhaps it would be more correct to say: there are three times: a present of things past, a present of things present, a present of things future." – Augustine of Hippo, *Confessions*.

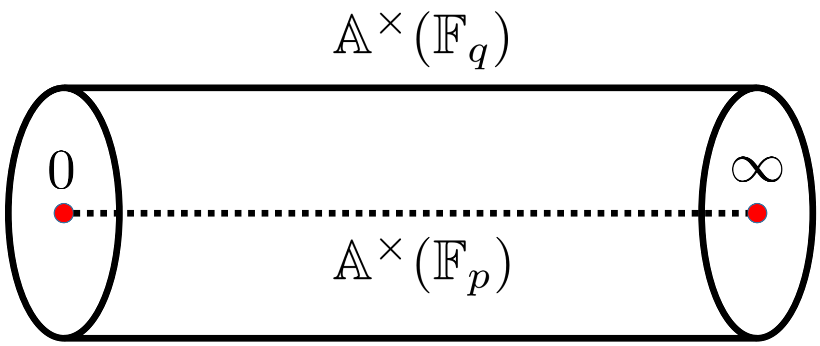

Figure 2: Depiction of the punctured affine line over $\mathbb{A}^\times(\mathbb{F}_q)$ and the Frobenius invariant subspace $\mathbb{A}^\times(\mathbb{F}_p)$. We view $\mathbb{A}^\times(\mathbb{F}_q)$ as specifying a cylinder with $\mathbb{A}^\times(\mathbb{F}_p)$ playing the role of a time coordinate. This picture is reinforced by the use of local Laurent series in which positive degree terms describe modes propagating to the future, as indicated by ∞, and the negative degree terms describe modes propagating to the past, as indicated by 0.

Our next task is to write down a mode expansion. This can be presented as a Laurent series with coefficients ϕ_n in $\mathbb{F}_q$:[18]

$$\phi = \sum_{n\in\mathbb{Z}} \phi_n u^n \in \mathbb{F}_q[u, u^{-1}]. \tag{5.4}$$

Here, we allow ourselves to consider arbitrarily high degree poles. At this point it is helpful to compare with the Laurent expansion we would write for a field $\phi_{\mathbb{C}}$ with support on $\mathbb{C}$, which we would also write as:

$$\phi_{\mathbb{C}} = \sum_{m,n\in\mathbb{Z}} \phi_{mn} z^m \overline{z}^n, \tag{5.5}$$

in the obvious notation. Here, we observe that we have dependence on a local coordinate z as well as its complex conjugate $\overline{z}$. In the case of a characteristic p space, however, the analog of complex conjugation just involves raising a variable to some prime power, so it is indeed appropriate to continue working in terms of an expansion in a single formal variable x.

Again turning to the characteristic zero setting, we observe that for a 2D theory specified on a cylinder with coordinate $z = \exp(\tau + i\sigma)$, the positive degree terms in the expansion

[18]But for now, we only allow a finite number of non-zero terms so that we can actually evaluate each character map on quantities such as the path integral phase factor $\exp(iS[\phi]/\hbar)$.

correspond to positive frequency modes of the expansion, while the negative degree terms in the expansion correlate with the negative frequency modes. Clearly, the same idea applies in our expansion of line (5.4), and this provides a notion of modes in the past versus the future. In more practical terms, we can focus on the subspace $\mathbb{F}_p \subset \mathbb{F}_q$, and pick a generator of the additive group $(\mathbb{F}_p, +)$. Then, the number of compositions by this additive subgroup defines a local notion of time ordering, which is in accord with our discussion of mode expansions.

We can now see how the above considerations can be generalized to rational maps $\phi : X \dashrightarrow Y$. Namely, we first seek out the local X_t dependence, and perform a mode expansion with respect to this coordinate. We then have a local definition of past and future, as specified by local terms which have negative or positive degree.

With this in place, we can introduce various notions of a Hilbert space of states. To motivate our proposal for the states in our system, we begin by recalling how things work in the standard path integral formalism in the Archimedean setting. To this end, we pick a fixed time slice, and specify the Hilbert space of states on this slice. Provided this is a Cauchy surface, we then have a well-posed time evolution problem, and so we can specify how states evolve in time in the Schrodinger picture, and one can of course equivalently work in the Heisenberg picture. In the path integral, the choice of field profile on a spatial slice amounts to prescribed boundary conditions, $\Phi(u_s)$ at some time slice specified, at say $t = 0$, and we use this to specify a state $|\Phi(u_s)\rangle$.

In the characteristic p setting we can also clearly speak of morphisms just defined on the spatial slice, namely given $\Phi : X_s \dashrightarrow Y$, we introduce a state $|\Phi : X_s \dashrightarrow Y\rangle$. Clearly, this provides (at least formally) a collection of states. On the other hand, since the notion of "time evolution" in the characteristic p setting is less clear (we return to this point shortly), it is also natural to consider a somewhat larger collection of morphisms which explicitly reference the time coordinate. Along these lines, consider the Laurent expansion for a field $\phi(t, u_s)$ near $t = 0$:

$$\phi(t, u_s) = \sum_{n\in\mathbb{Z}} \phi_n(u_s) t^n = \phi_{\leq}(u) + \phi_{>}(u) \tag{5.6}$$

where $\phi_{\leq}$ encodes the terms with $n \leq 0$ and $\phi_{>}$ encodes all terms with $n > 0$. For $n \leq 0$, each of the $\phi_n(u_s)$ can be viewed as specifying additional "initial state data". This motivates speaking of a somewhat broader notion of physical state which is locally defined by introducing:

$$|\phi_{\leq} : X \dashrightarrow Y\rangle, \tag{5.7}$$

namely we allow more general morphisms $\phi : X \dashrightarrow Y$, but in which we introduce an equivalence relation on all terms with positive degree terms in t. Thus, we form a vector space over $\mathbb{C}$ via the appropriate equivalence classes of morphisms $\phi_{\leq}$.

At this point, it is helpful to comment on the relation between the states just specified, and what happens if just consider the evaluation maps associated with these morphisms. Along these lines, suppose that we forget about all the data of the morphism. Then, we get a point given by all the possible values of $\phi(u)$ by considering the corresponding evaluation map $\mathrm{ev}_{x=u}(\phi(u)) \equiv \phi(x)$. Provided X and Y have a finite number of points, there is then a "small" states of states $|\phi(x)\rangle$ which also spans a finite dimensional vector space over $\mathbb{C}$.

Our discussion so far has mainly focused on constructing states, but to truly have a satisfactory notion of a quantum mechanical theory, we also need a notion of an inner product between states. We can calculate the overlap of field configurations in the "past and future" provided we make the additional assumption that there is a distinguished point $t_i \in X_t$ associated with an initial time, and $t_f \in X_t$ associated with a final time.[19] We can then prescribe fixed boundary conditions for our field at these two times, writing $\phi_{\leq}^{(i)} : X \dashrightarrow Y$ and $\phi_{\leq}^{(f)} : X \dashrightarrow Y$ for these two morphisms. Observe that we can truncate this discussion to just the non-polar terms to get a similar pairing for the states $|\Phi(u_s) : X_s \dashrightarrow Y\rangle$. With this in place, we can simply define an overlap of states by evaluating the path integral in characteristic p:

$$\langle \phi_{\geq}^{(f)} : X \dashrightarrow Y | \phi_{\leq}^{(i)} : X \dashrightarrow Y \rangle = \int\limits_{\phi_{\leq}^{(i)}:X\dashrightarrow Y}^{\phi_{\geq}^{(f)}:X\dashrightarrow Y} [d\phi] \ \exp(iS[\phi]/\hbar), \tag{5.8}$$

where here, the integral symbol is really an instruction to sum over all the rational morphisms $\phi : X \dashrightarrow Y$ with the prescribed boundary conditions at the marked points. We comment here that in the standard path integral, it is more common to just specify a "spatial morphism" via $\Phi_s : X_s \dashrightarrow Y$. Implicit in that approach, however, is that we have some notion of "taking a limit" of $\phi(t, u_s)$ onto a fixed value of t. The above prescription contains this, but can in principle also keep additional data, as tracked by the trajectory taken to $t \to t_i$ and $t \to t_f$.

The inner product just given is to be viewed as computing a suitable transition amplitude. Of course, we can consider the special case where the action $S[\phi] = 0$, and in this case we just get the "standard" inner product on a Hilbert space. In evaluating this, we are simply counting the number of interpolating morphisms which are compatible with the two boundary conditions set by $\phi^{(i)}$ and $\phi^{(f)}$, and so is typically either zero or infinite.[20] Note, however, that we can regulate the infinite values by truncating to polynomials of fixed degree (locally). Observe also that if we restrict to the "small" set of states where just record the actual images of the morphism $|\phi(x)\rangle$, then the inner product just collapses to the expect Kronecker delta, that is, $\langle a|b\rangle = \delta_{ab}$.

We are now ready to propose a definition of our physical Hilbert space compatible with our path integral formalism. In fact, there are various natural notions of a Hilbert space we have already encountered, and we shall refer to them as the "BIG Hilbert space" $\mathcal{H}_{\mathrm{BIG}}$, the

[19] In the punctured affine line example, $t_i = 0$ and $t_f = \infty$.

[20] This already happens in the Archimedean setting where we typically encounter Dirac delta functionals for overlaps of wave functionals.

"big Hilbert space" $\mathcal{H}_{\mathrm{big}}$, and the "small Hilbert space" $\mathcal{H}_{\mathrm{small}}$. The BIG Hilbert space is spanned by the collection of states defined by rational morphisms $\phi_{\leq} : X \dashrightarrow Y$ between two varieties, where we have an equivalence relation on any two morphisms which agree on the non-positive degree terms in the local Laurent expansion. The big Hilbert space is spanned by the collection of states defined by rational morphisms $\Phi : X_s \dashrightarrow Y$, where we only track the spatial dependence of a field. Finally, in the small Hilbert space, we forget both the temporal and spatial dependence, and just focus on the point set so obtained, i.e., the evaluation of $\phi(x)$. In the obvious notation, the Hilbert spaces are spanned by elements of the form:

$$|\phi_{\leq} : X \dashrightarrow Y\rangle \in \mathcal{H}_{\mathrm{BIG}} \tag{5.9}$$

$$|\Phi : X_s \dashrightarrow Y\rangle \in \mathcal{H}_{\mathrm{big}} \tag{5.10}$$

$$|\mathrm{ev}_{x_s=u_s}(\Phi(u_s))\rangle \in \mathcal{H}_{\mathrm{small}}. \tag{5.11}$$

We clearly have a set of a sequence of surjective projection maps:

$$\mathcal{H}_{\mathrm{BIG}} \to \mathcal{H}_{\mathrm{big}} \to \mathcal{H}_{\mathrm{small}}. \tag{5.12}$$

Observe also that with the spanning elements in place, we can then construct general linear combinations, with a norm topology dictated by the inner product.

Note that $\mathcal{H}_{\mathrm{BIG}}$ is always infinite dimensional (since it locally makes reference to polynomials of unbounded degree). Provided $\dim X_s > 0$, we also have $\mathcal{H}_{\mathrm{big}}$ is infinite-dimensional. On the other hand, the space $\mathcal{H}_{\mathrm{small}}$ is substantially smaller, consisting of a finite number of point set maps. Comparing with our discussion in section 3, we see that the Hilbert space for the lattice approximation of a physical system is more akin to $\mathcal{H}_{\mathrm{small}}$. An additional remark here is that in characteristic zero, one can often conflate these notions, but here we must tread more carefully.

We emphasize here that since our mode expansions do make reference to the past and future, there is also an implicit notion of (partial) time ordering, as referenced by the path integral. Returning to the general contours of our proposal where we indicated how this would work for the puncture affine line over $\mathbb{F}_q$, where the Frobenius invariant subspace $\mathbb{F}_p$ picks out a notion of a "time coordinate" with a past a 0 and a future at ∞. We can also introduce an on ordering on the affine line as induced from the fact that the additive group $(\mathbb{F}_p, +)$ can be generated by a single element. Of course, there can be more than one such generator, and this means that there is some implicit choice being made in such a time ordering.

In fact, one can make some choices to extend this notion of time ordering to the small Hilbert space. For example, we can speak of repeatedly applying a given operator such as $\exp\left(\frac{2\pi i}{p}\widehat{H}\right)$ with $\widehat{H}$ the Hamiltonian, and this composition rule builds up a local notion of past and future. This composition of maps defines a notion of time evolution, much as one would have in iterations of a discretized dynamical system. Of course, at a pragmatic level nothing stops us from just explicitly performing the requisite sum, so we have an operational rule for how to evaluate correlation functions.

5.1 Special Case: $\mathcal{H}_{\text{big}}(\mathbb{A}^1)$ vs $\mathcal{H}_{\text{small}}(\mathbb{A}^1)$

We now consider a special case where $X_s = \mathbb{A}^1$ and $Y = \mathbb{A}^1$ and compare the big and small Hilbert spaces $\mathcal{H}_{\text{big}}(\mathbb{A}^1)$ and $\mathcal{H}_{\text{small}}(\mathbb{A}^1)$. In this case, we are considering morphisms $\Phi : \mathbb{A}^1 \to \mathbb{A}^1$, and do not allow any further poles (i.e., polynomials rather than rational functions). Similar considerations apply for more general varieties because we can always glue together based on such affine patches anyway. Now, in this case, observe that our morphisms are really just given by polynomials of the form:

$$|\Phi : \mathbb{A}^1 \to \mathbb{A}^1\rangle = |\sum_{m\geq 0} \phi_m (u_s)^m\rangle = |\phi_0, ..., \phi_M, ...\rangle, \tag{5.13}$$

in the obvious notation. Observe in particular that we can build up an approximation of the big Hilbert space by just truncating to polynomials with degree M or less, and we refer to this as $\mathcal{H}_{\text{big}}^{(M)}$. There is clearly an inverse limit (see Appendix I for the relevant definitions):

$$\mathcal{H}_{\text{big}} = \lim_{\leftarrow} \mathcal{H}_{\text{big}}^{(M)}, \tag{5.14}$$

as specified by the inverse system defined by the appropriate surjections:

$$\mathcal{H}_{\text{small}} \simeq \mathcal{H}_{\text{big}}^{(0)} \leftarrow \mathcal{H}_{\text{big}}^{(1)} \leftarrow ... \leftarrow \mathcal{H}_{\text{big}}^{(M)} \leftarrow ..., \tag{5.15}$$

namely we just truncate to lower degree terms. In particular, we also have the isomorphism:

$$\mathcal{H}_{\text{big}}^{(M)} \simeq \mathcal{H}_{\text{small}}^{\otimes M}, \tag{5.16}$$

namely we just record all the "qudits" associated with these coefficients. Using this, we can specify a natural inner product by just using the one induced from the small Hilbert space. This is compatible with our more general treatment in terms of path integral boundary conditions since we have to specify how two spatially supported morphisms can interpolate anyway.[21] As already mentioned, since X and Y are varieties, this local construction clearly extends to more general morphisms. In this way we visualize both $\mathcal{H}_{\text{big}}$ and $\mathcal{H}_{\text{BIG}}$ as built up from suitable (inverse) limits of the small Hilbert space $\mathcal{H}_{\text{small}}$.

[21] In the characteristic setting it is often customary to use L^2 integrable functions as a natural space of wave-functionals, e.g., for $f, g \in \mathbb{L}^2(\mathbb{C})$ we can define our inner product via $\langle f|g\rangle = \int d\mu(x) f^*(x) g(x)$ in the obvious notation. In the characteristic p setting we can view the "matching of coefficients" as roughly mimicking comparison of Fourier modes, a them we return to later in section 8.

5.2 Operators

Much as we would in ordinary quantum mechanics, we can also specify operators which act on the big Hilbert space.[22] By way of example, we now fix a ground field $\mathbb{F}_q$ and assume we have some basis of physical scalar fields $\phi(t, u_s) = \phi(u)$ taking values in a vector bundle $\mathcal{E}$. Assume also that for each $x \in X$, we have a pairing $v_{ij} : \mathcal{V}_x \times \mathcal{V}_x \to \mathbb{F}_q$ which we can interpret as a "pairing matrix". This can be lifted in a natural way to a local function $v_{ij}(u)$. We can then construct a mapping of the form:

$$\mathcal{V} \times \mathcal{V} \to \mathbb{F}_q \to \mathbb{F}_p \tag{5.17}$$

$$(\phi, \beta) \mapsto v_{ij}\phi^i b^j \mapsto \mathrm{Tr}(v_{ij}\phi^i b^j), \tag{5.18}$$

where in the last line we used the standard trace map for finite fields.

Just as in ordinary quantum mechanics, we wish to focus on operators which are unitary. In our setting, the analogous treatment amounts to the condition that the operators appearing in various exponentiated quantities are really valued in $\mathbb{F}_p$, and not just some characteristic p field. This restriction can be implemented in the above by focusing on $\mathbb{F}_p$-valued exponents, namely by working with operators such as:

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}(\mathrm{ev}_{u=x} v_{ij}(u)\phi^i(u)b^j(u))\right). \tag{5.19}$$

Of course, in a standard physical theory, we expect there to be some redundancy in our basis of states, since for example, we have a notion of time evolution of Cauchy slices. Something similar can be arranged in the characteristic p setting because we can also introduce a Legendre transform of our Lagrangian, and consequently define a Hamiltonian operator. Doing so, however, requires fixing a notion of a local time coordinate, and so we must make some further assumptions on the geometry of X, as we noted above. Making these choices, we can speak of operators specified at a fixed time in terms of the local Laurent expansion of a field (as well as its time derivative) near the local time coordinate.

With this in place, we now explain how operators act on states of the big Hilbert space given by spatial morphisms $|\Phi : X_s \dashrightarrow \mathcal{V}\rangle$.[23] With respect to a fixed value of the time (i.e., where we expand around) $t = t_*$, we can consider the evaluation of a field $\Phi(u_s) = \phi(t_*, u_s)$ as valued in $\mathcal{V}$, and a conjugate momentum $\Pi_j(u_s) = \partial_t\phi_j(t_*, u_s)$ valued in $\mathcal{V}^*$, the dual vector space defined using the canonical pairing v_{ij}. In this way, we get corresponding "spatial morphisms", and this notion can be extended to the BIG Hilbert space by simply tracking the appropriate polar terms.

Now, while it is of course tempting to consider the direct action of operators $\widehat{\Phi}$ and $\widehat{\Pi}$ on

[22]One can similarly build up a collection of operators for the BIG Hilbert space, but we leave this implicit in what follows.

[23]We specialize from a general target space Y to one such as $\mathcal{V}$.

our states, this is rather ill-defined, for the primary reason that these fields make reference to coefficients in a finite field, whereas our Hilbert space of states is defined over the standard complex numbers $\mathbb{C}$. That being said, characters from finite fields to $\mathbb{C}$ still make sense, so we can introduce an "exponentiated" version of such operators. It is actually simpler to begin with the operator associated with momentum, where exponentiation produces a translation. On the big Hilbert space we introduce an operation R_b as follows:

$$R_b \equiv \exp\left(\frac{2\pi i}{p}\widehat{\Pi}_j(u_s)b^j(u_s)\right), \tag{5.20}$$

with the action on the spanning states of the big Hilbert space given by:

$$R_b|\Phi : X_s \dashrightarrow \mathcal{V}\rangle = |(\Phi + b) : X_s \dashrightarrow \mathcal{V}\rangle, \tag{5.21}$$

namely we have a "translation map" which shifts the spatial morphism Φ to $\Phi + b$.

Consider next the action of $\widehat{\Phi}$ on a state. Again, while one might wish to consider the direct action of $\widehat{\Phi}$ on states, the associated "eigenvalue" would be a polynomial in a characteristic p field. One might also be tempted to consider the exponentiated action of $\widehat{\Phi}$, much as we did for translations in equation (5.21). Observe that in this case, however, we expect to return an overall complex phase, whereas what we would instead get is an exponentiated polynomial in a finite field. The resolution of this issue requires us to make an additional choice as associated with evaluating at a particular point of X_s, which we generically refer to as x_s.[24] With this caveats in place, we can define an operator via its action on the spatial morphism states:

$$T_a(x_s) \equiv \exp\left(\frac{2\pi i}{p}\mathrm{ev}_{u_s=x_s}(\widehat{\Phi}_j(u_s)a^j(u_s))\right) \tag{5.22}$$

with the action on the spanning states of the big Hilbert space given by:

$$T_a(x_s)|\Phi : X_s \dashrightarrow \mathcal{V}\rangle = \exp\left(\frac{2\pi i}{p}\mathrm{Tr}((\mathrm{ev}_{u_s=x_s}(\widehat{\Phi}_j(u_s)a^j(u_s))\right)|\Phi : X_s \dashrightarrow \mathcal{V}\rangle. \tag{5.23}$$

Functorially, we can also consider a more general operator $T_a(\bullet)$, where we leave the evaluation on a given x_s implicit.

Summarizing, then, we have introduced two families of operators R_b and $T_a(\bullet)$. Whereas $T_a(\bullet)$ requires a further marked point to produce an actual map on the Hilbert space, R_b does not require this choice. Observe also that if we project onto the small Hilbert space, these distinctions no longer make an appearance. Continuing in this way, it should now be clear that even though our physical fields are valued in characteristic p varieties, we are still able to make sense of a physical Hilbert space over the characteristic zero field $\mathbb{C}$.

[24]If ϕ is not valued in $\mathbb{F}_p$, one must consider a further composition with an evaluation map $Y \to \mathbb{F}_p$ to produce a sensible answer.

6 Quantum Error Correction

In the previous sections we showed that there is a path integral in characteristic p, and that we can use this to build up various notions of a Hilbert space of states. With this in place, we now turn the discussion around once more and show that our formulation of "physics in characteristic p" has a natural interpretation in terms of structures found in the study of quantum error correcting codes. Along these lines, we first recall that there is a well-known use for finite fields as a means to construct examples of classical and quantum error correcting codes. In this way one can view the present section as an attempt to build up the previously encountered physical structures from a purely information theoretic starting point.

More precisely, in this section we make use of the structure of classical codes generated from algebraic curves over finite fields, as well as their extension to quantum error correcting codes. To keep the discussion streamlined, we have deferred some standard definitions to Appendix F which contains additional details. See figure 3 for a depiction of encoding via schemes over finite fields.

To frame the discussion to follow, recall that in information theory, one is often concerned with the transmission of information across a (possibly noisy) channel. There are various ways to generalize the classical notions of information transmission to the quantum setting, depending on whether one is interested in sending classical / quantum information over a classical / quantum channel. For our purposes, it is enough to speak of an "input Hilbert space" $\mathcal{H}_A$ and an "output Hilbert space" $\mathcal{H}_B$. We can then consider (bounded) linear operators on each Hilbert space $\mathrm{Lin}(\mathcal{H}_A)$ and $\mathrm{Lin}(\mathcal{H}_B)$, which importantly, includes all possible density matrices of a mixed quantum state. A linear map between these spaces:

$$\mathcal{Q} : \mathrm{Lin}(\mathcal{H}_A) \to \mathrm{Lin}(\mathcal{H}_B), \tag{6.1}$$

specifies a quantum channel when it is trace preserving.[25]

Roughly speaking, we can view our field theory path integral as summing over quantum error correcting codes, with X and Y respectively viewed as the "source" and "target". Classically speaking, we view each morphism $\phi : X \dashrightarrow Y$ as specifying a word, i.e., each point in $x_i \in X$ serves as an input, and each target space point $\phi(x_i) \in Y$ serves as the encoded word. For this to carry over to the quantum setting, it is natural to introduce auxiliary spaces X' and Y' and corresponding Hilbert spaces of morphisms $\mathcal{H}_A = \mathcal{H}_{X' \dashrightarrow X}$ and $\mathcal{H}_B = \mathcal{H}_{Y' \dashrightarrow Y}$, where we can restrict to the case of $\mathcal{H}_{\mathrm{BIG}}$, $\mathcal{H}_{\mathrm{big}}$ or $\mathcal{H}_{\mathrm{small}}$, as necessary. Then, we can speak of channels as in line (6.1). A morphism $\phi : X \dashrightarrow Y$ can then be used to build an error correcting code.[26]

[25] One can weaken this to trace non-decreasing, since given such a map one can build a trace preserving map by suitable amendments.

[26] One might also ask whether there are any canonical choices for forming a quantum channel. For our present purposes we remain agnostic as to whether there is a preferred quantum channel since we leave

Of course, this begs the question as to how to identify "natural" choices of X' and Y'. There are at least two choices which appear in many other contexts. For one, we can view both X and Y as the target space of a string theory, with X' and Y' as the corresponding worldsheet, much as we discussed in section 3:

$$X' \dashrightarrow X \dashrightarrow Y \dashleftarrow Y'. \tag{6.2}$$

Alternatively, we can also set $Y' = X$ and just deal with the composition of morphisms of the form:

$$X' \dashrightarrow X \dashrightarrow Y. \tag{6.3}$$

In this case, one could view X' as a worldsheet, or some other construct. In any case, we shall assume the existence of the suitable spaces such that we can speak of a corresponding quantum channel. With this in place, we can then turn to the construction of suitable error correcting codes.

In what follows, we mainly focus on the details of how this works for the small Hilbert space. This is a special case of the big Hilbert space where the spatial geometry X_s is trivial. One can of course extend our considerations to this broader setting since the operations of lines (5.21) and (5.23) specify generalized qudit operations. In the case of the small Hilbert space construction, the details of X' and Y' matter less anyway because we simply keep the "evaluation points" of the associated morphisms.

As a first step in this direction, suppose we consider a scalar field theory, where we take our spacetime X to be $\mathbb{P}^1(\mathbb{F}_q)$, the projective line over the finite field $\mathbb{F}_q$, and our physical field is a rational map $\phi : X \dashrightarrow Y$ with Y the affine line $\mathbb{A}^1(\mathbb{F}_q)$. We mark the "point at infinity" in X and specify a prescribed pole structure at this location. Given this, we can interpret X as a single timelike direction to construct a Hilbert space of states. This requires us to also specify a notion of a canonical pairing v_{ij} which we implicitly use to raise and lower indices. In the Heisenberg picture, we label these states as $|\phi\rangle$, where ϕ is interpreted as taking values in $\mathbb{F}_q$, where we introduce at some fixed time t_* an operator Φ^i and its conjugate momentum $\Pi_j = \partial_t \phi_j|_{t=t_*}$, with the properties that for $a, b \in \mathbb{F}_q$ we have:

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}a^i\Phi^j)\right)|\phi\rangle = \exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}a^i\phi^j)\right)|\phi\rangle \tag{6.4}$$

$$\exp\left(\frac{2\pi i}{p}b^j\Pi_j\right)|\phi\rangle = |\phi + b\rangle\,, \tag{6.5}$$

so we recognize that these are building up standard qudit error operations which we denote as:

$$E_{ab} = T_a R_b \tag{6.6}$$

the source of "errors" in transmission generic. In the physical setting natural choices include (perhaps as in [111, 112]) a suitable notion of coarse graining / renormalization (which does have a characteristic p analog), but we leave the study of this question for future work.

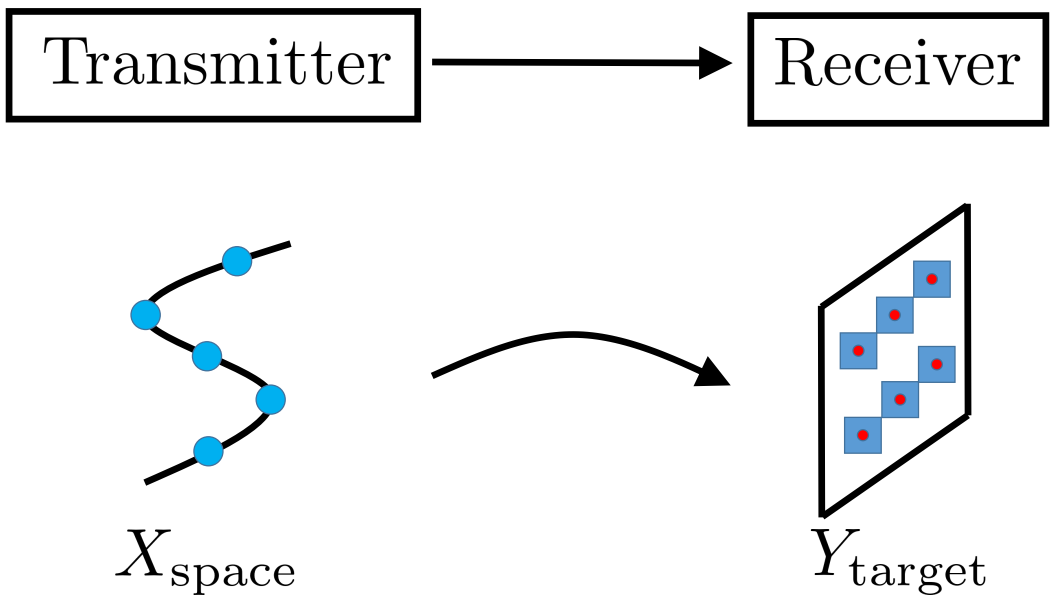

Figure 3: Depiction of how a spatial geometry furnishes a code, both in the classical and quantum setting. The encoding of information is achieved via maps from a transmitter (source) to a receiver (target space).

with:

$$T_a = \exp\left(\frac{2\pi i}{p}\mathrm{Tr}(v_{ij}a^i\Phi^j)\right), \quad R_b = \exp\left(\frac{2\pi i}{p}b^j\Pi_j\right), \tag{6.7}$$

so in other words, we get a single qudit in the case of a quantum mechanical system defined in this way.

We now start generalizing to more involved field theoretic examples. We first illustrate show how the associated morphisms automatically produce classical error correcting codes, and then explain how this extends to quantum error correcting codes.

Along these lines, we now take our spacetime X to be given by $X = X_{\text{time}} \times X_{\text{space}}$, where again we assume X_{time} is a projective line, and X_{space} is now taken to be a smooth projective curve over the ground field $\mathbb{F}_q$. In this case, we again take Y to be an affine line, so we might as well view ϕ as an element of the line bundle $\mathcal{L}(G)$, with G a divisor indicating the "points at infinity" of X. From what we have said above, we should also view the a's and b's appearing in our discussion above as elements of the line bundle $\mathcal{L}(G)$. We can now build a set of linear codes by considering n distinct points $P_1, ..., P_n$ of X_{space} as well as the divisor $D = P_1 + ... + P_n$. Evaluating at these points, we get the classical linear code $C_{\mathcal{L}}(D, G)$. Of course, in our path integral prescription we perform this evaluation, but in a slightly more involved away, first constructing an evaluation map to the action, and then mapping this to a character of our finite field. The point remains, however, that at least in this simplified setting, each of our physical field configurations can be viewed as an element of the Riemann-Roch space.

Having seen how classical linear codes naturally emerge from this setting, we can ask about the construction of quantum stabilizer codes. As reviewed in Appendix F below line (F.34), one of the standard results is to actually specify an $\mathbb{F}_p$ linear subspace $C \subset \mathbb{F}_q^{2n}$ which is self-orthogonal with respect to the standard symplectic pairing. In the present context, we consider the simplifying situation where C is actually an $\mathbb{F}_q$-linear subspace of dimension k. The precise idea is to specify the duals to the vectors a and b as elements $(\Phi(P_l); \Pi(P_l))$ evaluated at the points in X_s specified by the divisor $D = P_1 + ... + P_n$. Here, the index $l = 1, ..., n$ runs over the evaluation points, and we have left implicit the vector index. The main thing we need to establish is that our phase space builds a self-orthogonal space with respect to the standard symplectic pairing. For ease of exposition, we assume that we can change basis so that the v_{ij} appearing earlier is just the identity matrix and we assume the standard relation:

$$\Pi = \partial_t \Phi, \tag{6.8}$$

in the obvious notation. Now, given two elements $(\Phi; \Pi)$ and $(\Phi'; \Pi')$ of C, we observe that the symplectic pairing is:

$$(\Phi; \Pi) *_s (\Phi'; \Pi') = \mathrm{Tr}(\Phi \cdot \Pi' - \Pi' \cdot \Phi). \tag{6.9}$$

But, via equation (6.8), we observe that $\Pi' = \partial_t \Phi'$ and $\Pi = \partial_t \Phi$. By inspection of equation (6.9), the condition that the code is self-orthogonal with respect to $*_s$ now follows. Treating these classical evaluation points as the span of possible values of the a and b appearing in the CSS construction [113, 114], the theorem of reference [115] stated below line (F.34) now gives us an $[[n, n-k, d(C^{(s)} \backslash C)]]_q$ quantum stabilizer code. To aid the reader, recall that in such a code, the subscript "q" indicates we work with the "alphabet" $\mathbb{F}_q$, the first parameter indicates we are using n total qudits to encode the information provided by a $n-k$-dimensional qudits. The last argument is a "distance", indicating roughly speaking how many errors can be corrected. We are here introducing a symplectic phase space $C^{(s)}$, and specifying a Lagrangian subspace C (namely, a polarization). See Appendix F for further discussion.

We note that from the perspective of symplectic geometry in characteristic zero, all that we have done is exploit the appearance of a Lagrangian submanifold in the symplectic phase space, i.e., a middle dimensional subspace which provides a canonical split between positions Φ and conjugate momenta $\Pi = \partial_t \Phi$. The characteristic p analog of this statements provides us with our construction of a self-orthogonal linear subspace.

The above statements also generalize to the case where we work with more general sorts of classical codes. As mentioned in Appendix F near line (F.17), we can also consider situations where we have a stable vector bundle E over a curve X. We will return to examples of such structures in section 15. Recall from our discussion in Appendix F, a rank r vector bundle allows us to specify a code subspace in $\mathbb{F}_Q^n$, where $Q = q^r$. While this is

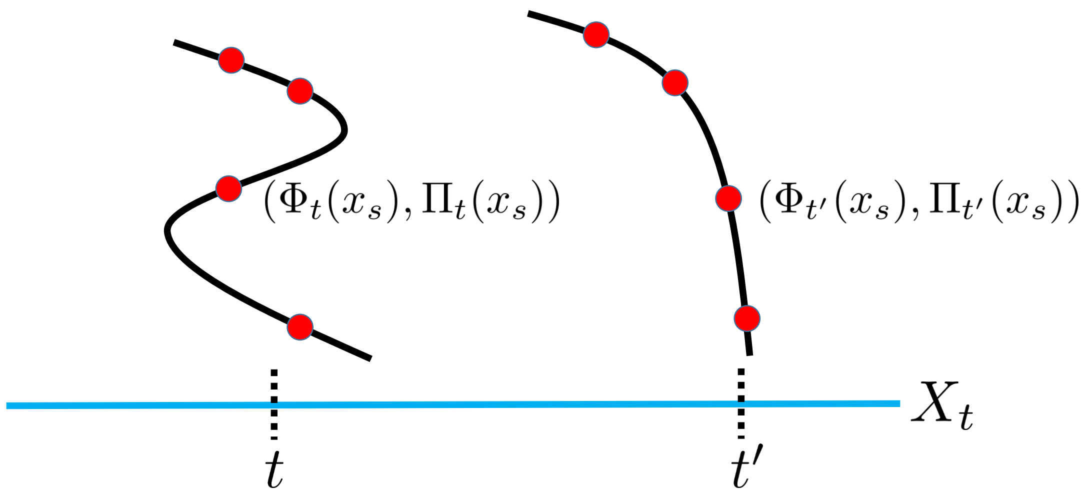

Figure 4: Depiction of the motion of the trajectory of a field $\Phi_t(x_s)$ and its conjugate momentum $\Pi_t(x_s)$ in phase space as a function of the time coordinate in X_{time}. This phase space structure provides a self-orthogonal space with respect to the symplectic pairing $*_s$, and thus can generate a quantum stabilizer code.

not always an $\mathbb{F}_Q$ linear space, it is $\mathbb{F}_q$ linear. Consequently, we can use the same sort of "phase space argument" used above in the case of line bundles (where $r = 1$), we then get an $[[n, n-k, d]]_Q$ quantum stabilizer code.

From the above considerations, we thus see a different physical interpretation of our construction of field theories in characteristic p. In particular, we can also view our path integral as performing a sum over possible classical codewords (after composing with evaluation maps) and the resulting eigenspaces generated by the resulting quantum error correcting codes define a basis of states in a physical Hilbert space. Observe that in constructing this Hilbert space, even if we have roughly n physical points (as dictated by the order of the divisor D in X_s), the actual information content is instead "delocalized" across several points.

6.1 Quasi-Locality

There is one more layer of abstraction which naturally fits with our discussion, and suggests how the structure of locality emerges from a somewhat more primitive construct. At a rough level, we view this as stating that our formalism for defining path integrals in characteristic p amounts to specifying a topos of quantum stabilizer codes. Indeed, there is some suggestive overlap with potential uses of topoi in physics discussed in references [18–23] which might be interesting to develop further. That being said, we will not attempt a match with the considerations found there.

To set the stage, we recall that one of the awkward features of algebraic geometry in general is the rather course nature of the Zariski topology. In characteristic zero, one can often supplement this by treating various spaces as real or complex analytic, but in working with finite fields there is always an intrinsic discretization to the resulting system, a point which we are actually attempting to exploit. Nevertheless, there is a sufficiently rich notion of topology we can introduce which allows one to construct non-trivial cohomology theories. We review some features of the resulting topologies in Appendix E. The main point is that to get a suitable notion of "coverings by open sets" it is important to emphasize more the collections of morphisms to a given scheme.

In our context, we have already seen that there is a sense in which the spacetime X as well as the target space Y can be equipped with suitable topoi. From the perspective of coding theory, we have also interpreted this as a general transmission problem in information theory. From an ambitious standpoint, one might view the associated topologies as defining a construct even more primitive than a Hilbert space, the latter only appearing after further processing in the language of quantum error correcting codes.

Of course, if we ever wish to return to the world of observation, we must somehow find a path back from characteristic p geometry to the more familiar terrain of physics in characteristic zero. We turn to these issues later in part III.

7 Scale Entanglement and an Emergent Bulk

In this section we show that there is a close analogy to renormalization in the characteristic p setting. Moreover, we use this to to show that much as in the proposal of reference for building up a bulk from a tensor network [8–10], this can be interpreted in terms of a higher-dimensional "gravity dual". The structure we introduce is reminiscent of similar observations made in the p-adic setting, as in [11–13], and we return to this theme later in sections 18 and 19.

To begin, let us first explain the sense in which we even have "energy scales" in the characteristic p setting. It is enough to illustrate the point by considering states of $\mathcal{H}_{\text{big}}$ in the special case where the space $X_s = \mathbb{A}^1(\mathbb{F}_q)$, the affine line. Similar considerations hold for more general choices of X_s, as well as for states of $\mathcal{H}_{\text{BIG}}$.

Working with the $\mathcal{H}_{\text{big}}$ Hilbert space, we have states labelled by rational morphisms $\Phi : \mathbb{A}^1(\mathbb{F}_q) \dashrightarrow \mathbb{A}^1(\mathbb{F}_q)$, and for ease of exposition we focus on morphisms as captured by polynomials in $\mathbb{F}_q[u]$:[27]

$$\Phi(u) = \sum_{m=0}^{M} \Phi_m u^m. \tag{7.1}$$

Now, in the Archimedean setting, we can interpret the u as functions such as $\exp(\tau + i\sigma))$, with $\tau + i\sigma$ a local (holomorphic) coordinate on a 2D worldsheet. Higher powers of u^m for m large can thus be viewed as higher frequency / shorter distance modes in the expansion. Note also that this clearly extends to rational morphisms with support on the punctured affine line $\mathbb{A}^{\times}(\mathbb{F}_q) = \text{Spec}\,(\mathbb{F}_q[u, u^{-1}])$. One can also develop a more "Lorentz covariant" treatment which focuses on the space of all rational morphisms $X \dashrightarrow Y$ by considering a similar truncation in the degrees of the numerator and denominator of a rational function.

Putting these generalizations aside, observe that if we restrict to polynomials of degree at most M, then we are just filling out states in $\mathcal{H}_{\text{big}}^{(M)}$, and as we already remarked in section 5, there is an isomorphism $\mathcal{H}_{\text{big}}^{(M)} \simeq \mathcal{H}_{\text{small}}^{\otimes M}$, with each morphism just labelled by a tuple $|\Phi_0, ..., \Phi_M\rangle$. This tensor product automatically comes with a hierarchical ordering since we can reference high scale / low scale coefficients in each Φ_m:

$$\mathcal{H}_{\text{big}}^{(M)} \simeq \mathcal{H}_{\text{small}}^{(m=0)} \otimes \mathcal{H}_{\text{small}}^{(m=1)} \otimes ... \otimes \mathcal{H}_{\text{small}}^{(m=M-1)} \otimes \mathcal{H}_{\text{small}}^{(m=M)}. \tag{7.2}$$

We also have various natural maps from $\mathcal{H}_{\text{big}}^{(M)}$ to $\mathcal{H}_{\text{big}}^{(M-1)}$. Perhaps the simplest is just the "projection map":

$$\mathcal{H}_{\text{big}}^{(M)} \to \mathcal{H}_{\text{big}}^{(M-1)} \tag{7.3}$$

$$|\Phi_0, ..., \Phi_M\rangle \mapsto |\Phi_0, ..., \Phi_{M-1}\rangle\,, \tag{7.4}$$

[27]The extension to rational morphisms of the form $P(u)/Q(u)$ is similar, we simply truncate the degrees of P and Q, and "coarse grain" by restricting these degrees further. One can also extend to power series expressions, as is appropriate when taking the inverse limit to reach $\mathcal{H}_{\text{big}}$.

in which we simply forget the last entry in the $(M+1)$-tuple. Note also that we can also embed $\mathcal{H}_{\text{big}}^{(M-1)} \to \mathcal{H}_{\text{big}}^{(M)}$ by just "padding by a zero" on $\Phi_M = 0$. More general isometries are clearly also possible in which we simply consider any linear map $W : \mathcal{H}_{\text{big}}^{(M-1)} \to \mathcal{H}_{\text{big}}^{(M)}$ such that[28] $W^\dagger W = \mathbb{I}_{\mathcal{H}_{\text{big}}^{(M-1)}}$ and $WW^\dagger$ is a projection map.

Now, given this structure, and especially the hierarchical ordering of tensor product factors in line (7.2), there is a clear sense in which we can consider the entanglement of states across different scales, much as in [116] (see also [111]). To illustrate, consider a density matrix constructed from states in $\mathcal{H}_{\text{big}}^{(M)}$. Then, performing a partial trace over $\mathcal{H}_{\text{small}}^{(m=M)}$, we reach a (possibly mixed) state of $\mathcal{H}_{\text{big}}^{(M-1)}$. Suppose then that we are given a (possibly mixed) state $\rho \in \text{Lin}(\mathcal{H}_{\text{big}}^{(M)})$. Following the standard steps in MERA / DMRG (see e.g., [117]), we can perform a partial trace over $\mathcal{H}_{\text{small}}^{(m=M)}$ to get a reduced density matrix $\overline{\rho}$. We can then follow this by an isometry to reembed in $\mathcal{H}_{\text{big}}^{(M)}$, i.e. we map $\overline{\rho} \mapsto W\overline{\rho}W^\dagger$. Acting by a suitable "disentangling" unitary operator $U : \mathcal{H}_{\text{big}}^{(M)} \to \mathcal{H}_{\text{big}}^{(M)}$, we can map $\overline{\rho} \mapsto W\overline{\rho}W^\dagger \mapsto UW\overline{\rho}W^\dagger U^\dagger$, we get a sequence of coarse grained density matrices. The main condition we aim to impose is that the various correlation functions $\text{Tr}(\rho\mathcal{O}_1 \ldots \mathcal{O}_k)$ for operators (computed using our path integral) remain invariant under this coarse graining step, which in turn constrains our choice of W and U. In this sense, we have an analog of renormalization / scale entanglement in the characteristic p setting.

Observe that since M is now arbitrary, we see that our discussion extends (by taking an inverse limit) to the entire big Hilbert space $\mathcal{H}_{\text{big}}$, and similar considerations also apply for $\mathcal{H}_{\text{BIG}}$ as well as the entire collection of rational morphisms $X \dashrightarrow Y$ used to set up our path integral in the first place.

There is also a precise sense in which the procedure just outlined builds up a tree-like structure which can be visualized as constructing a "bulk geometry". This is a recurring theme in recent holographic studies, and this includes, recent investigations of p-adic analogs of the AdS/CFT correspondence, a topic we turn to in sections 18 and 19. To see how this comes about when working with polynomials over $\mathbb{F}_q$, we recall that each higher degree term amounts to a further decision, informed by the lower degree terms. Along these lines, given a polynomial:

$$\Phi(u) = \sum_{m=0}^{M} \Phi_m u^m, \tag{7.5}$$

we begin by specifying one of q initial choices for Φ_0, the coefficient of the u^0 term. Having made this choice, we move on to a choice of q possible values for Φ_1. This builds up a tree-like structure. Indeed, introducing an initial node for the monomial u^0, we can attach to it q distinct nodes indicating the q possible coefficients $\omega_0, \ldots, \omega_{q-1} \in \mathbb{F}_q$ for the coefficient Φ_0. Then, for each of these nodes we attach q more nodes to the right to indicate a further choice. Each element of $\mathbb{F}_q[u]$ can then be presented as a finite length path beginning at

[28] The hermitian conjugate is implicitly specified by the choice of inner product, something we discussed in section 5.

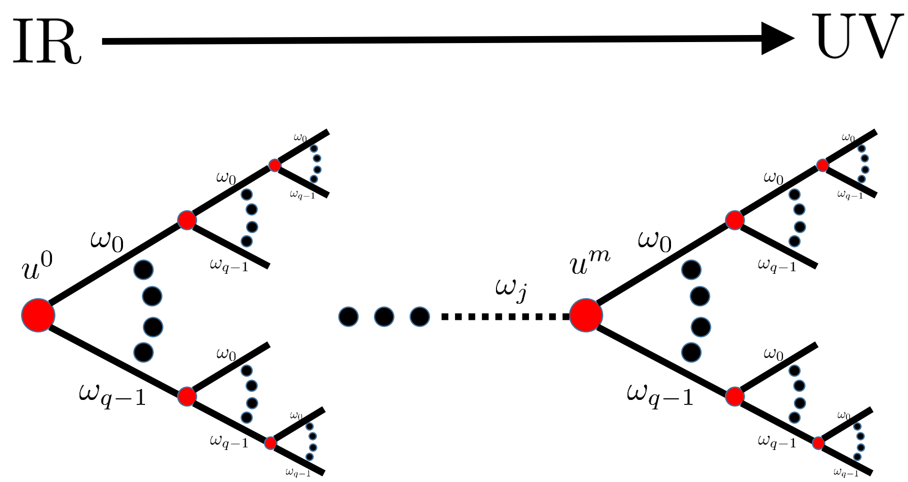

Figure 5: Depiction of the tree-like structure built up from polynomials over $\mathbb{F}_q[u]$. Visualizing this as a decision tree, we read from left to right possible coefficients of degree u^m terms. Similar considerations hold for coordinate rings / functions fields of more general varieties.

the u^0 node and terminating after some number of steps. See figure 5 for a depiction of this tree-like structure.[29]

This construction makes special reference to a starting node. One might ask whether this persists in other cases. For example, working over the punctured affine line $\mathbb{A}^\times$, the relevant polynomials are represented as $\mathbb{F}_q[u, u^{-1}]$ which we can present as:

$$\Phi(u) = \sum_{|m| \leq M} \Phi_m u^m, \tag{7.6}$$

namely we allow both positive and negative degree terms. In this case, the notion of coarse graining is a bit more subtle, but it is still appropriate to view the low values of $|m|$ as specifying the IR and the large values of $|m|$ as specifying the UV. In this case as well, then, we see that there is still a preferred "initial node".[30]

[29]In the p-adic setting it is convenient to view the links as specifying the coefficient of the "node to the right" rather than the "node to the left", but it is just a mild change of convention.

[30]It is tempting to generalize a bit further by constructing the Bruhat-Tits tree $PGL_2(\mathbb{F}_q(u))/PGL_2(\mathbb{F}_q[u])$ which has a number of formal similarities to the p-adic case (see sections 18, 19 and 20). In the present case, however, it is less clear to us whether the resulting structure admits an interpretation in terms of coarse graining.

7.1 Examples of Scale Entangled States

Before closing this section, let us give some explicit examples of states which exhibit scale entanglement. Some of these examples are well-known in other contexts, but the additional structure of polynomials over finite fields at least provides some novel ways to build / interpret such examples.

As a warmup, we can start with the pure GHZ-state on $M+1$ qudits:

$$\rho_{\mathrm{GHZ}(M+1)} = \left|\mathrm{GHZ}(M+1)\right\rangle \left\langle \mathrm{GHZ}(M+1)\right|, \tag{7.7}$$

with:

$$\left|\mathrm{GHZ}(M+1)\right\rangle = \frac{1}{\sqrt{2}}\underbrace{\left|0,...,0\right\rangle}_{M+1} + \frac{1}{\sqrt{2}}\underbrace{\left|1,...,1\right\rangle}_{M+1} \in \mathcal{H}_{\mathrm{big}}^{(M)}. \tag{7.8}$$

Note that this makes sense over any choice of ground field $\mathbb{F}_q$. As polynomials over $\mathbb{F}_q$, these states are particularly natural, since:

$$\underbrace{\left|0,...,0\right\rangle}_{M+1} = \left|0 \in \mathbb{F}_q[u]\right\rangle, \tag{7.9}$$

$$\underbrace{\left|1,...,1\right\rangle}_{M+1} = \left|1 + ... + u^M \in \mathbb{F}_q[u]\right\rangle. \tag{7.10}$$

The resulting partial trace is a mixed state on M qudits (built from states of $\mathcal{H}_{\mathrm{big}}^{(M-1)}$)

$$\overline{\rho}_{\mathrm{GHZ}(M+1)} = \mathrm{Tr}_{\left(\mathcal{H}_{\mathrm{small}}^{(m=M)}\right)}\rho_{\mathrm{GHZ}(M+1)} = \frac{1}{2}\underbrace{\left|0,...,0\right\rangle}_{M}\underbrace{\left\langle 0,...,0\right|}_{M} + \frac{1}{2}\underbrace{\left|1,...,1\right\rangle}_{M}\underbrace{\left\langle 1,...,1\right|}_{M}. \tag{7.11}$$

Similarly, the W-state is given by an appropriate sum over states defined by monomials:

$$\left|\mathrm{W}(M+1)\right\rangle = \frac{1}{\sqrt{M+1}}\sum_{m=0}^{M}\left|u^m \in \mathbb{F}_q[u]\right\rangle \in \mathcal{H}_{\mathrm{big}}^{(M)}, \tag{7.12}$$

and this also produces a mixed state after performing a partial trace over $\mathcal{H}_{\mathrm{small}}^{(m=M)}$.

One way to build new examples is to start with an irreducible polynomial $\Phi(u) \in \mathbb{F}_q$ which has a splitting field $K/\mathbb{F}_q$. Labelling the roots as $\alpha_1, ..., \alpha_l$, one can consider the states $\left|u - \alpha_i \in K[u]\right\rangle$. There is a natural action of $\mathrm{Gal}(K/\mathbb{F}_q)$ on this space of states, as given by permutation of the roots. This extends to a linear action on all of $\mathcal{H}_{\mathrm{big}}(K)$ (i.e., the big Hilbert space involving $K[u]$ rather than $\mathbb{F}_q[u]$). In terms of the qudit basis, each of these is represented as $\left|\alpha_j, 1, 0, ..., 0, ...\right\rangle \in \mathcal{H}_{\mathrm{big}}(K)$. We now build a pure state $\rho = \left|\Psi\right\rangle\left\langle\Psi\right|$ which has scale entanglement and which is also invariant under this Galois group action. To this end, introduce the state:

$$|\Psi\rangle = \frac{1}{\sqrt{|\mathrm{Gal}(K/\mathbb{F}_q)|}} \sum_i |u - \alpha_i \in K[u]\rangle \in \mathcal{H}_{\mathrm{big}}. \tag{7.13}$$

Performing a partial trace over the complement of $\mathcal{H}^{(m=0)}_{\mathrm{small}}$, observe that:

$$\overline{\rho} = \mathrm{Tr}_{\left(\overline{\mathcal{H}^{(m=0)}_{\mathrm{small}}}\right)} \rho = \frac{1}{|\mathrm{Gal}(K/\mathbb{F}_q)|} \sum_i |\alpha_i\rangle \langle\alpha_i| \in \mathrm{Lin}(\mathcal{H}^{(m=0)}_{\mathrm{small}}), \tag{7.14}$$

and the entanglement entropy is just:

$$-\,\mathrm{Tr}\overline{\rho} \log \overline{\rho} = \log |\mathrm{Gal}(K/\mathbb{F}_q)|\,. \tag{7.15}$$

Similar considerations clearly hold for more general irreducible polynomials and field extensions.

8 Mode Expansions and Feynman Diagrams

In the previous sections we proceeded to increasing levels of abstraction to set up our general formalism for how to define a physical system in characteristic p. In this section we show how to perform some explicit computations in "practice," showing how to implement some explicit mode expansions and perform loop corrections in this setting.

For ease of exposition, we first focus on the simplest non-trivial case in which we have a bosonic field theory, as specified by morphisms of the punctured affine line to the affine line:

$$\phi : \mathbb{A}^{\times} \to \mathbb{A}, \tag{8.1}$$

as in our discussion of line (5.3). We assume the ground field is $\mathbb{F}_q$, and so we can present a physical field as an element of:

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n u^n \in \mathbb{F}_q[u, u^{-1}] \subset \mathbb{F}_q[[u, u^{-1}]], \tag{8.2}$$

namely only a finite number of the terms are ever non-zero.

One of the main reasons to introduce a mode expansion of this sort is that it can help in analyzing the structure of interaction terms. In characteristic zero, we have the standard properties of Fourier transforms, but we might ask whether any of this structure carries over to the present setting. To aid us in our analysis, it is helpful to consider the function:

$$f(l) = \sum_{x \in \mathbb{F}_q^{\times}} x^l, \tag{8.3}$$

for l an integer. Unless l divides $(q-1)$, f is identically zero, and when l divides $(q-1)$, $f(l) = -1$, namely:

$$f(l) = \begin{cases} 0 & \text{if} \quad l \nmid (q-1) \\ -1 & \text{if} \quad l | (q-1) \end{cases}, \tag{8.4}$$

To see why, we recall that the multiplicative group $\mathbb{F}_q^{\times}$ is actually a cyclic group of order $q-1$. Letting g denote a generator of this cyclic group, we can now write:

$$f(l) = \sum_{x \in \mathbb{F}_q^{\times}} x^l = \sum_{i=1}^{q-1} g^{il}. \tag{8.5}$$

Consider next multiplying $f(l)$ by g^l. This yields:

$$g^l f(l) = \sum_{i=1}^{q-1} g^{l(i+1)} = \sum_{i=2}^{q} g^{li} = f(l), \tag{8.6}$$

where the final equality follows from the fact that we are still summing over all the elements of $\mathbb{F}_q^\times$. Now, since g is a generator of $\mathbb{F}_q^\times$, we know that if l divides $q-1$, then $g^l = 1$. On the other hand, if l does not divide $q-1$, then $g^l \neq 1$. For equation (8.6) to hold when $l \nmid (q-1)$ means $f(l) = 0$, establishing the claim.

When l divides $q-1$, we observe that for all elements $x \in \mathbb{F}_q$, we have the relation $x^q = x$, and for all non-zero elements we have $x^{q-1} = 1$. Consequently, we have, in this case:

$$\text{If } l|(q-1) : f(l) = \sum_{x \in \mathbb{F}_q^\times} x^l = q - 1 = -1, \tag{8.7}$$

since we are working in characteristic p.

Defining a mode expansion is now a relatively straightforward affair provided we can capture all the values where f is non-zero. Given two expansions:

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n u^n \in \mathbb{F}_q[u, u^{-1}] \tag{8.8}$$

$$\psi = \sum_{n \in \mathbb{Z}} \psi_n u^n \in \mathbb{F}_q[u, u^{-1}], \tag{8.9}$$

we observe that:

$$\sum_{x \in \mathbb{F}_q^\times} \mathrm{ev}_{u=x}(\phi\psi) = \sum_{m.n} -\phi_m \psi_n \widehat{\delta}_{m+n}, \tag{8.10}$$

where we have introduced a modified Kronecker delta:

$$\widehat{\delta}_l = \delta\left(l = 0 \bmod (q-1)\right). \tag{8.11}$$

The usual statement of momentum conservation would have enforced $l = 0$, but in the setting of finite fields, we must relax this condition and only enforce it mod $q - 1$.

We can also work out a Fourier decomposition of a quartic interaction term:

$$\sum_{x \in \mathbb{F}_q^\times} \mathrm{ev}_{u=x}\phi^4 = \sum_{x \in \mathbb{F}_q^\times} \mathrm{ev}_{u=x}\phi^4 = \sum_{m,n,r,s} -\phi_m\phi_n\phi_r\phi_s \widehat{\delta}_{m+n+r+s}. \tag{8.12}$$

We can work out a similar mode expansion for the kinetic terms. It is actually technically simpler (and well-motivated) to use the derivative $D_u \equiv u\partial_u$:

$$\mathrm{Kin} = \sum_{x \in \mathbb{F}_q^\times} \kappa \mathrm{ev}_{u=x}(D_u\phi)^2 = \sum_{m,n} \kappa \left(-mn\phi_m\phi_n\widehat{\delta}_{m+n}\right). \tag{8.13}$$

We say that that the use of $D_u = u\partial_u$ is “well-motivated” because in “cylindrical coordinates” this leads to a simpler structure for the resulting mode expansions, and the same holds true here as well.[31]

At this point, we see that there is a complication coming from the fact that we only appear to have momentum conservation mod $q-1$. To study this issue more closely, we observe that we can now construct the formal sums:

$$\Phi_m = (... + (m-(q-1))\phi_{m-(q-1)} + m\phi_m + (m+(q-1))\phi_{m+(q-1)} + ...). \tag{8.16}$$

Of course, the behavior of such an infinite sum is not fully defined since we do not have a notion of convergence of this sum. We can, however, consider a regulated version of the sum centered on m:

$$\Phi_m^{(M,N)} = \sum_{\alpha=-M}^{\alpha=+N} (m+\alpha(q-1))\phi_{m+\alpha(q-1)} = \sum_{\alpha=-M}^{\alpha=+N} (m-\alpha)\phi_{m+\alpha(q-1)}. \tag{8.17}$$

In the final equality we used the fact that we are working in characteristic p and $q=0$ mod p. We shall often specialize to the case $M=N$.

Now, to understand the structure of correlation functions in this theory, it is helpful to adopt an alternative notation. With this in mind, we view our modes $\phi_{m+\alpha(q-1)}$ as specified by a matrix ϕ_m^α where the range of possible values are:

$$\phi_m^\alpha \text{ modes: } m \in \{1,...,q-1\} \quad \text{and} \quad \alpha \in \mathbb{Z}. \tag{8.18}$$

For brevity, we shall reference the α index via the vector notation $\overrightarrow{\phi}_m$ with the standard dot product operation. In particular, we write:

$$\sum_{\alpha=-M}^{\alpha=+N} (m-\alpha)\phi_{m+\alpha(q-1)} = \overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m \tag{8.19}$$

---

[31] As a brief aside, the use of mode expansions allows us to address one item regarding the presentation of the kinetic term, and the usage (or lack thereof) of "integration by parts", where one equates terms such as $\partial_u\phi\partial_u\phi$ with $-\phi\partial_u^2\phi$, as is customary in the characteristic zero setting. Here, we see that in matching the modes in the characteristic p setting, we can make the substitutions:

$$(D_u\phi)^2 = (u\partial_u\phi)^2 \sim -\phi u\partial_u(u\partial_u\phi). \tag{8.14}$$

The sign flip in this case has to do with matching the expansion modes; on the lefthand side we have terms such as $mn\phi_m\phi_n\widehat{\delta}_{n+m}$, whereas on the righthand side we have terms such as $n^2\phi_m\phi_n\widehat{\delta}_{m+n}$. By the same token, we can also perform an "integration by parts" for other choices of kinetic terms. For example, we have:

$$(\partial_u\phi)^2 \sim -\phi u\partial_u(u^{-1}(\partial_u\phi)). \tag{8.15}$$

Observe that in this case, we need to insert a factor of u^{-1} to properly match the mode expansions on the two sides. This is acceptable provided we work on a space such as the punctured affine line $\mathbb{A}^\times$, since then, the inverse always exists. This also serves to illustrate some of the additional benefits of using the derivative $D_u = u\partial_u$. Some additional aspects of mode expansions and their connection to eigenfunctions of D_u are discussed in section 9 and 10.

where:

$$\mu_m^\alpha = (m-\alpha), \text{ for } m = 1, ..., q-1. \tag{8.20}$$

We comment here that later on, we will often let the index m go "out of range" by being negative. In such situations, we have $\phi_{-m}^\alpha = \phi_{q-1-m}^\alpha$ and $\mu_{-m}^\alpha = \mu_{q-1-m}^\alpha$ when $m = 1, ..., q-2$, and for $m = q-1$ (the "zero mode") we instead have: $\phi_{-(q-1)}^\alpha = \phi_{q-1}^\alpha$ and $\mu_{-(q-1)}^\alpha = \mu_{q-1}^\alpha$.

In this notation, the kinetic energy density of line (8.13) can be written as:

$$\text{Kin} = \sum_{x \in \mathbb{F}_q^\times} \kappa \text{ev}_{u=x}(u\partial_u \phi)^2 = \sum_{m,n} -\kappa mn \phi_m \phi_n \widehat{\delta}_{m+n} = \sum_{m=1}^{q-1} -\kappa(\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m)(\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\phi}_{-m}), \tag{8.21}$$

namely, we now have a finite sum over the momentum index $m = 1, ..., q-1$, but we still have a regulated infinite sum over the α index. Here, we have introduced a slight abuse of notation, since "$-m$" is not valued in $\{1, ..., q-1\}$, but working mod $q-1$, the meaning is clear, and we shall find it helpful to permit this abuse in what follows. By inspection, we note that this has the form of an outer product in the alpha index, and as such, the number of propagating modes is vastly smaller than one might have a priori suspected.

8.1 Correlation Functions in a Gaussian Model

To illustrate this in more detail, let us now consider the evaluation of some correlation functions in the Gaussian model. For ease of exposition, we assume that we have a 1D model (i.e., just time derivatives) and that the ground field is $\mathbb{F}_p$. The generalization to more dimensions and other finite fields is straightforward enough. So, for now we specialize to the field $\mathbb{F}_p$ and take our action to be:

$$S[\phi] = \sum_{x \in \mathbb{F}_p^\times} \kappa \text{ev}_{u=x}(D_u \phi)^2 = \sum_{m=1}^{p-1} -\kappa(\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m)(\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\phi}_{-m}). \tag{8.22}$$

See Appendix G for a sample calculation of the partition function for a related model.

Our aim here will be to calculate some example correlation functions. We shall mainly focus on correlation functions involving generalized vertex operators of the form:

$$\mathcal{O}_J = \exp\left(\frac{2\pi i}{p} \sum_{m=1}^{p-1} \overrightarrow{J}_m \cdot \overrightarrow{\phi}_m\right) \tag{8.23}$$

for some specific choice of source vector $\overrightarrow{J}_{-m}$. In what follow we shall make some simplifying choices for $\overrightarrow{J}_{-m}$ since we are mainly interested in understanding how the propagating degrees of freedom in the model interact with one another (i.e., the non-zero modes).

In carrying out the evaluation of correlation functions, it suffices to study the single expectation value:

$$\langle \mathcal{O}_J \rangle = \frac{\sum\limits_{\phi} \exp(iS[\phi]/\hbar)\mathcal{O}_J}{\sum\limits_{\phi} \exp(iS/\hbar)}, \tag{8.24}$$

since we also have:

$$\exp(iS[\phi]/\hbar)\mathcal{O}_J = \exp(iS[\phi, J]/\hbar), \tag{8.25}$$

with $S[\phi, J]$ the action in the presence of a background source:

$$S[\phi, J] = \sum_{m=1}^{p-1} -\kappa(\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m)(\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\phi}_{-m}) + \overrightarrow{J}_m \cdot \overrightarrow{\phi}_m. \tag{8.26}$$

Now, compared with the characteristic zero case, the treatment of zero modes is somewhat more delicate. Indeed, in our kinetic term we can already see an issue because only the linear combinations and $\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m$ actually propagates. To analyze this structure, we therefore begin by picking a convenient basis with which to expand our various modes.

Along these lines, our plan will be to simplify the path integral over a single $\overrightarrow{\phi}_m$ mode. By inspection of the expression $\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m$, we see that there is a sense in which the only component of $\overrightarrow{\phi}_m$ which actually enters our path integral sum is associated with the components of $\overrightarrow{\phi}_m$ parallel to $\overrightarrow{\mu}_m$. In more detail, we begin by fixing a basis for our vector space $\overrightarrow{w}^{(0)}, \overrightarrow{w}^{(1)}, ..., \overrightarrow{w}^{(l)}, ...$, where formally speaking the index l extends over all the integers (there is no penalty in re-indexing so that all indices here are positive integers). We can always choose the $\overrightarrow{w}^{(l)}$ such that $\overrightarrow{w}^{(l)} \cdot \overrightarrow{w}^{(l)} \neq 0$, with $\overrightarrow{w}^{(k)} \cdot \overrightarrow{w}^{(l)} = 0$ for $k \neq l$. To analyze the structure of non-zero propagating degrees of freedom we split up our discussion into two cases:

$$\text{Case 1:} \quad \overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m \neq 0 \tag{8.27}$$

$$\text{Case 2:} \quad \overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m = 0 \tag{8.28}$$

Indeed, the main complication in projecting onto appropriate subspaces is that now, we can have "null vectors".

8.1.1 Case 1: $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m \neq 0$

Suppose first that $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m \neq 0$. In this case, it is helpful to set $\overrightarrow{\mu}_m = \overrightarrow{w}_m^{(0)} \equiv \overrightarrow{v}_m^{(0)}$ and consider the collection of vectors:

$$\overrightarrow{v}_m^{(l)} = \overrightarrow{w}_m^{(l)} - \left(\frac{\overrightarrow{w}_m^{(0)} \cdot \overrightarrow{w}_m^{(l)}}{\overrightarrow{w}_m^{(0)} \cdot \overrightarrow{w}_m^{(0)}} \right) \overrightarrow{w}_m^{(0)} \quad \text{for } l \neq 0. \tag{8.29}$$

Observe that the $\overrightarrow{v}_m^{(l)}$ still form a basis and that $\overrightarrow{\mu}_m \cdot \overrightarrow{v}_m^{(l)} = 0$ for all $l \neq 0$. We can expand in terms of:

$$\overrightarrow{\phi}_m = a_m \overrightarrow{v}_m^{(0)} + \sum_{l \neq 0} a_m^{(l)} \overrightarrow{v}_m^{(l)} \tag{8.30}$$

with:

$$\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m = a_m. \tag{8.31}$$

The range of values for a_{-m} are the p distinct values in the finite field $\mathbb{F}_p$. These are the propogating degrees of freedom in the model.

8.1.2 Case 2: $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m = 0$

Suppose next that $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m = 0$. In this case, we cannot resort to the analog of equation (**??**) since it would involve division by $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m$. Instead, we show that we can limit ourselves to a single $\overrightarrow{v}_m^{(0)}$ which is not orthogonal to $\overrightarrow{\mu}_m$. To see why, suppose without loss of generality that $\overrightarrow{\mu}_m \cdot \overrightarrow{w}_m^{(0)} \neq 0$, with $\overrightarrow{w}_m^{(0)} \cdot \overrightarrow{w}_m^{(l)} = 0$ for $l \neq 0$. Set $\overrightarrow{v}_m^{(0)} = \overrightarrow{w}_m^{(0)}$. For $l \neq 0$, we instead set:

$$\overrightarrow{v}_m^{(l)} = \overrightarrow{w}_m^{(l)} - \left(\frac{\overrightarrow{\mu}_m \cdot \overrightarrow{w}_m^{(l)}}{\overrightarrow{\mu}_m \cdot \overrightarrow{w}_m^{(0)}} \right) \overrightarrow{w}_m^{(0)} \quad \text{for} \quad l \neq 0. \tag{8.32}$$

By inspection, we have that $\overrightarrow{\mu}_m \cdot \overrightarrow{w}_m^{(l)} = 0$ for $l \neq 0$. For future use, we also observe that $\overrightarrow{v}_m^{(0)} \cdot \overrightarrow{v}_m^{(l)} = 0$ for $l \neq 0$ since $\overrightarrow{w}_m^{(0)} \cdot \overrightarrow{w}_m^{(l)} = 0$ for $l \neq 0$. So in this case there is an expasion of the form:

$$\overrightarrow{\phi}_m = a_m \overrightarrow{\nu}_m^{(0)} + \sum_{l \neq 0} a_m^{(l)} \overrightarrow{\nu}_m^{(l)}, \tag{8.33}$$

in which:

$$\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m = a_m. \tag{8.34}$$

The range of values for a_m are the p distinct values in the finite field $\mathbb{F}_p$. These are the propogating degrees of freedom in the model.

8.1.3 Computing Correlators

Returning to our task of computing $\langle \mathcal{O}_J \rangle$, we now make the assumption that the defining sources for these operators satisfy:

$$\overrightarrow{J}_m = j_m \overrightarrow{v}_m^{(0)} \tag{8.35}$$

This is all that is really required, because only some of the $\overrightarrow{\phi}$'s can propagate anyway. Indeed, observe that:

$$\overrightarrow{J}_m \cdot \overrightarrow{\phi}_m = \left(j_m \overrightarrow{v}_m^{(0)}\right) \cdot \left(a_m \overrightarrow{\nu}_m^{(0)} + \sum_{l \neq 0} a_m^{(l)} \overrightarrow{\nu}_m^{(l)}\right) \tag{8.36}$$

$$= j_m a_m \left(\overrightarrow{\nu}_m^{(0)} \cdot \overrightarrow{\nu}_m^{(0)}\right), \tag{8.37}$$

where in the above, we used the fact that $\overrightarrow{v}_m^{(0)} \cdot \overrightarrow{v}_m^{(l)} = 0$ for $l \neq 0$.

We can then write the operator $\mathcal{O}_J$ as:

$$\mathcal{O}_J = \exp\left(\frac{2\pi i}{p} \sum_{m=1}^{p-1} j_m a_m \left(\overrightarrow{\nu}_m^{(0)} \cdot \overrightarrow{\nu}_m^{(0)}\right)\right) \tag{8.38}$$

Turning next to $\exp(iS[\phi]/\hbar)\mathcal{O}_J$, we have:

$$\exp(iS[\phi]/\hbar)\mathcal{O}_J = \exp\left(\frac{2\pi i}{p} \sum_{m=1}^{p-1} \left(\begin{array}{l} -\kappa a_{-m} a_m \left(\overrightarrow{\nu}_m^{(0)} \cdot \overrightarrow{\nu}_m^{(0)}\right) \left(\overrightarrow{\nu}_{-m}^{(0)} \cdot \overrightarrow{\nu}_{-m}^{(0)}\right) \\ +\frac{1}{2} j_m a_m \left(\overrightarrow{\nu}_m^{(0)} \cdot \overrightarrow{\nu}_m^{(0)}\right) \\ +\frac{1}{2} j_{-m} a_{-m} \left(\overrightarrow{\nu}_{-m}^{(0)} \cdot \overrightarrow{\nu}_{-m}^{(0)}\right) \end{array}\right)\right). \tag{8.39}$$

Completing the square by making the substitution:

$$a_m \mapsto a_m - \frac{1}{2\kappa} \frac{j_{-m}}{\overrightarrow{\nu}_m^{(0)} \cdot \overrightarrow{\nu}_m^{(0)}}, \tag{8.40}$$

and the expectation value of $\mathcal{O}_J$ is therefore:

$$\langle \mathcal{O}_J \rangle = \exp\left(\frac{2\pi i}{p} \sum_{m=1}^{p-1} -\frac{1}{4\kappa} j_m j_{-m}\right) \tag{8.41}$$

$$= \exp\left(\frac{2\pi i}{p} \sum_{m=1}^{p-1} -\frac{1}{4\kappa} \left(\frac{\overrightarrow{J}_m \cdot \overrightarrow{\nu}_m^{(0)}}{\overrightarrow{\nu}_m^{(0)} \cdot \overrightarrow{\nu}_m^{(0)}}\right) \left(\frac{\overrightarrow{J}_{-m} \cdot \overrightarrow{\nu}_{-m}^{(0)}}{\overrightarrow{\nu}_{-m}^{(0)} \cdot \overrightarrow{\nu}_{-m}^{(0)}}\right)\right). \tag{8.42}$$

With this in place, we can in principle evaluate any number of correlation functions involving operators $\langle \mathcal{O}_{\omega_1} ... \mathcal{O}_{\omega_k} \rangle$, simplying by setting $J = \omega_1 + ... + \omega_k$.

Let us now attempt to make contact with the standard analysis in characteristic zero. So long as we remember to exponentiate back to form characters, we can formally write down the two-point function for modes of the scalar field theory by formally differentiating $\langle \mathcal{O}_J \rangle$ with respect to J_m^{α} and J_{-m}^{β} and then setting $\overrightarrow{J}_m$ to zero. In particular, for modes where we have $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m \neq 0$, we can set $\overrightarrow{\nu}_m^{(0)} = \overrightarrow{\mu}_m$, and we get:

$$\left\langle \phi_m^\alpha \phi_{-m}^\beta \right\rangle = -\frac{p}{2\pi i}\frac{1}{2\kappa}\left(\frac{\mu_m^\alpha}{\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m}\right)\left(\frac{\mu_{-m}^\beta}{\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\mu}_{-m}}\right). \tag{8.43}$$

To proceed further, we now need to put the "out of range" values of $-m$ back in range. Since the modes are being indexed mod $p-1$, this means $\mu_{-m}^\beta = \mu_{p-1-m}^\beta = (p-1-m-\beta)$. It is also helpful to explicitly evaluate the regulated sum $\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m$ centered on m:

$$\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m = \sum_{\alpha=-N}^{\alpha=+N}(m-\alpha)^2 = m^2(2N+1) + -2mN(N+1) + \frac{N(N+1)(2N+1)}{3}. \tag{8.44}$$

Observe that when $N = p-1 \bmod p$, the regulated sum collapses to $-m^2$. This seems to be a natural choice which matches with characteristic zero expectations, and so we adopt it in what follows. Summarizing then, in this regulator we can now make the substitution:

$$\overrightarrow{\mu}_m \cdot \overrightarrow{\mu}_m = m^2. \tag{8.45}$$

Consider next the regulated dot product for $\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\mu}_{-m}$. According to our indexing prescription, we have to regulate $\overrightarrow{\mu}_{p-1-m} \cdot \overrightarrow{\mu}_{p-1-m}$. Now, if simply apply the formula from equation (8.45),one might want to simply equate this with $-(m+1)^2$. Observe, however, that this choice of regulated dot product would vanish when $m = -1 = p-1$, which in turn would signal the presence of a singularity at specific momenta in the correlator of line (8.43). In light of this, it seems more appropriate to instead consider an adjusted window for the regulated sum in (8.44) where we write:

$$\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\mu}_{-m} = \sum_{\alpha=-N-1}^{\alpha=+N-1}(p-1-m-\alpha)^2 = \sum_{\alpha=-N}^{\alpha=+N}(-m-\alpha)^2 = m^2, \tag{8.46}$$

which would not produce a spurious singularity in equation (8.43). Taking this at face value, the two point function then evaluates to:

We can now evaluate our two point functions:

$$\left\langle \phi_m^\alpha \phi_{-m}^\beta \right\rangle = \frac{p}{2\pi i}\frac{1}{2\kappa}\frac{(m-\alpha)(m+\beta+1)}{m^4}. \tag{8.47}$$

In the special case where we set $\alpha = 0$ and $\beta = -1$, we also have

$$\left\langle \phi_m^{\alpha=0} \phi_{-m}^{\beta=-1} \right\rangle = \frac{p}{2\pi i}\frac{1}{2\kappa}\frac{1}{m^2}, \tag{8.48}$$

which is immediately recognizable compared with its characteristic zero analog.

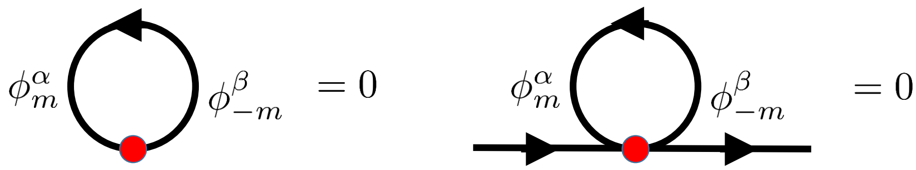

Figure 6: Depiction of some one loop diagrams. The internal loop involves a sum over modes ϕ_m^α and ϕ_{-m}^β, with m indexing a finite number of momentum modes and α, β a formally infinite number of terms. The diagrams are evaluated by summing over all admissible values of α, β and m. With respect to a well-motivated regulated sum over α and β, the one-loop bubble diagram (left) vanishes, as well as the one-loop correction to the two-point function in scalar ϕ^4 theory. In both cases, we assume $p \neq 2, 3$.

8.1.4 Bubbles and Zero Point Energies

Another curiosity of this setting is the structure of various loop corrections. Returning to our expression for the two-point function (to avoid clutter we now set $\kappa = 1/2$):

$$\left\langle \phi_m^\alpha \phi_{-m}^\beta \right\rangle = \frac{p}{2\pi i} \frac{1}{2\kappa} \frac{(m-\alpha)(m+\beta+1)}{m^4}. \tag{8.49}$$

We can ask about the evaluation of this sum over the various momenta. On general grounds, the summation over the m will generically involve expressions such as:

$$\sum_{m=1}^{p-1} \frac{1}{m^2}, \; \sum_{m=1}^{p-1} \frac{1}{m^3}, \; \sum_{m=1}^{p-1} \frac{1}{m^4}. \tag{8.50}$$

But, when p is a prime greater than 5, our previous discussion near line (8.4) established that such sums actually vanishes! So, the corresponding bubble diagrams / contributions to the cosmological constant of the system actually vanish. See figure 6 for a depiction of a one loop bubble diagram.

Similar considerations hold in the higher-dimensional setting as well. While we leave a full treatment to future work, consider the extension of our sum on momenta such as provided by a k-component vector with entries m^a for $a = 1, ..., k$. Assume also a diagonal non-degenerate quadratic form h_{ab}, and we reference the "norm" as:

$$\|m\|^2 = h_{ab} m^a m^b. \tag{8.51}$$

We now show via induction on k that the sum over $1/\|m\|^2$ for all non-zero values of $\|m\|^2$

also vanishes.

We have already established the claim for $k = 1$, so suppose it holds for some value of k. To indicate this, we append a subscript to our norm-squared, writing $\|m\|_k^2$ in the obvious way. Consider next extending our quadratic form by an additional variable. Then, we can write:

$$\sum_{\|m\|_{k+1}^2 \neq 0} \frac{1}{\|m\|_{k+1}^2} = \sum_{\|m\|_{k+1}^2 \neq 0} \frac{1}{h_{k+1,k+1}(m^{(k+1)})^2 + \|m\|_k^2}. \tag{8.52}$$

We now break up the sum into those contributions where $m^{(k+1)} = 0$, and those for which $m^{(k+1)} \neq 0$. In the contribution from $m^{(k+1)} = 0$, though, we are just performing the sum over $1/\|m\|_k^2$, and via our inductive step we already know this vanishes. So, it is enough to assume that $m^{(k+1)} \neq 0$. In this case, we are free to divide by this quantity to write:

$$\sum_{\|m\|_{k+1}^2 \neq 0} \frac{1}{\|m\|_{k+1}^2} = \sum_{m^{(k+1)}=1}^{p-1} \frac{1}{\left(m^{(k+1)}\right)^2} \sum_{\|m\|_k^2 + \left(m^{(k+1)}\right)^2 h_{k+1,k+1} \neq 0} \frac{1}{h_{k+1,k+1} + \frac{1}{\left(m^{(k+1)}\right)^2} \|m\|_k^2}, \tag{8.53}$$

in the obvious notation. Next, rescale all the entries of the remaining k-component vector by $m^{(k+1)}$. We can do this because we are summing over all such k-component vectors anyway. Then, we can write our sum as:

$$\sum_{\|m\|_{k+1}^2 \neq 0} \frac{1}{\|m\|_{k+1}^2} = \sum_{m^{(k+1)}=1}^{p-1} \frac{1}{\left(m^{(k+1)}\right)^2} \sum_{\|m\|_k^2 + h_{k+1,k+1} \neq 0} \frac{1}{h_{k+1,k+1} + \|m\|_k^2}. \tag{8.54}$$

But now the sum over $1/\left(m^{(k+1)}\right)^2$ vanishes, and the claim is established. Clearly, a similar set of manipulations holds in similar situations.

By the same token, a number of other sums of this sort also identically vanish. For example, we observe that:

$$\sum_{\|m\|^2 \neq 0} \frac{1}{(\|m\|^2)^l} = 0 \quad \text{if} \quad (p-1) \nmid 2l \tag{8.55}$$

$$\sum_{\|m\|^2 \neq 0} \frac{1}{(\|m\|^2)^l} = -1 \quad \text{if} \quad (p-1) | 2l. \tag{8.56}$$

where in the above, we have assumed $p \neq 2, 3$.

So, we see that the loop corrections in such theories are still non-trivial but that in many cases there are signicant simplifications.

8.2 Loop Corrections in ϕ^4 Theory

To further investigate the structure of loop corrections in this setting, we now turn the canonical example of massless ϕ^4 theory in characteristic $p \neq 2, 3, 5$. Our Lagrangian is given by:

$$L = T - V \tag{8.57}$$

where the kinetic energy functional is:[32]

$$T = \frac{1}{2}\left((D_t\phi)^2 - (D_x\phi)^2 - (D_y\phi)^2 - (D_z\phi)^2\right) \quad \text{and} \quad V = \frac{\lambda}{4!}\phi^4, \tag{8.58}$$

in the obvious notation. We denote the four-momentum as m^a.

Consider first the structure of the two-point function. In the free field approximation the two-point function in momentum space is:

$$\left\langle \phi^{\alpha}_{m}\phi^{\beta}_{-m}\right\rangle_{\text{free}} = -\frac{p}{2\pi i}\frac{(m^a - \alpha^a)(\pi_a - m_a - \beta_a)}{(m^a m_a)^2}, \tag{8.59}$$

where $\pi^a = (p-1, p-1, p-1, p-1) = (-1, -1, -1, -1)$ is a four-component vector associated with the "negative momentum". Let us calculate the one-loop correction to this propagator. Observe, however, that this leads to loop corrections of the form:

$$\sum_{m^a m_a \neq 0} \frac{1}{(m^a m_a)^l} = 0, \tag{8.60}$$

and so the one-loop diagram actually contributes nothing.

Consider next the structure of the two-point function. In the free field approximation the four-point function in momentum space is:

$$\left\langle \phi^{\alpha}_{m}\phi^{\beta}_{n}\phi^{\gamma}_{r}\phi^{\delta}_{s}\right\rangle_{\text{free}} = -\frac{2\pi i}{p}\lambda\widehat{\delta}_{m+n+r+s}. \tag{8.61}$$

Again, we emphasize that this expression is really a shorthand for insertion into exponentiated physical fields. The reason for this is simply that this expression with $1/p$ is meaningless in characteristic p. Additionally, we remark here that the sense in which we can speak of a sensible perturbation theory rests on expanding (in characteristic zero) the ratio λ/p. So, if p is indeed a large prime, such an expansion does still seem to make sense, so long as we treat all operator correlation functions as characters valued in $\mathbb{C}$. Another point of view is that we can think of these correlation functions as specifying a formal power series in the coupling constants of the Lagrangian (much as we would in the standard characteristic zero setting). Then, we are free to truncate this power series and evaluate at some finite degree values. Interpreting this as the logarithm of a suitably defined character with values in $\mathbb{C}$,

[32]Here, t, x, y, z should be viewed as specifying local coordinates, i.e., not specific points in the variety.

the convergence (or lack thereof) of the power series then follows from the induced metric properties on $\mathbb{C}$. See figure 6 for a depiction of the one loop correction to the two-point function.

Let us now turn to the one-loop correction to the coupling λ. By inspection, this boils down to evaluation of the following sort over momentum sum encountered in equation (8.55). The relevant momentum sum is obtained by setting all external momenta to zero. An example of such a contribution is:

$$\mathcal{I} = \sum_{m^a m_a \neq 0} \sum_{\alpha,\beta,\gamma,\delta} \frac{(m^a - \alpha^a)(\pi_a - m_a - \beta_a)(m^b - \gamma^b)(\pi_b - m_b - \delta_a)}{(m^a m_a)^2 \, (m^b m_b)^2} = 0, \tag{8.62}$$

where the last equality follows from similar considerations to those already presented.

More generally, we can ask about the loop corrections to four-point scattering amplitudes, with n a momentum transfer scale. An example of such a contribution is:

$$\mathcal{I}(n) = \sum_{m^a m_a \neq 0} \sum_{\alpha,\beta,\gamma,\delta} \frac{(m^a - \alpha^a)(\pi_a - m_a - \beta_a)(n^b + m^b - \gamma^b)(\pi_b - n_b - m_b - \delta_a)}{(m^a m_a)^2 \, ((n^b + m^b)(n_b + m_b))^2}, \tag{8.63}$$

where we observe that setting $n = 0$ reduces us to the case of equation (8.62). In this case, the external momenta are non-zero, which in turn makes the contribution to various loop corrections more intricate. A priori, it is not clear to us that this loop correction vanishes. This is all to the good because it indicates that perhaps an analog of the optical theorem persists in characteristic p.

8.3 Effective Potential and Higher Dimension Operators

More generally, we can study the structure of higher dimension operators in such field theories. Again by way of example, we focus on the effective potential for the quantum field theory of a a single scalar field ϕ.

Now, in characteristic zero, we often view a potential of a low energy effective as specified by a formal power series of the form:

$$V(\phi) = \sum_m V_m \phi^m \in \mathbb{R}[[\phi]]. \tag{8.64}$$

We would like to understand the analogous structure in characteristic p. We view the effective potential as a formal power series with (for ϕ a morphism of a variety over $\mathbb{F}_p$) coefficients in $\mathbb{F}_p$. All of this can be extended to other finite fields by including suitable Frobenius conjugate terms, but for ease of exposition we focus on the simplest non-trivial case:

$$V(\phi) = \sum_m V_m \phi^m \in \mathbb{F}_p[[\phi]]. \tag{8.65}$$

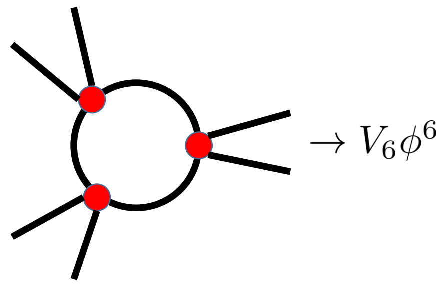

Figure 7: Example of a one loop contribution to a sextic interaction $V_6\phi^6$ in the effective potential of a physical theory as generated in ϕ^4 theory. In the case where the field is massless, this correction vanishes in the characteristic p setting, but is generically non-zero when the field is massive.

As a first comment, such corrections can indeed be generated by radiative corrections. The example of ϕ^4 theory is sufficient to illustrate the main points. On general grounds, we expect to generate all possible terms in the effective action compatible with the $\mathbb{Z}/2\mathbb{Z}$ symmetry associated with $\phi \mapsto -\phi$. As an illustrative example, consider possible loop corrections which might generate a ϕ^6 term (see figure 7). We observe that at least in the case where ϕ is exactly massless, our previous discussion of loop corrections shows that this contribution actually vanishes. When ϕ is massive (i.e., there is a quadratic term in the effective potential) then we do not observe such an exact cancellation, so in general the structure of the effective potential is indeed non-trivial.

Physically, we are accustomed to thinking of the values of V at different points in spacetime $X_{\text{spacetime}}$. Doing so, we observe the feature that for any truncated form of our effective potential, V will take values in a finite field. Another way to capture this feature is to observe that for $\phi \in \mathbb{F}_p$, we have the identity $\phi^p = \phi$. This is very much in tune with the structure of the Hilbert space discussed in section 5.

Indeed, if we evaluate the physical potential $V(\phi)$ at a given point of $X_{\text{spacetime}}$, then high degree terms in this polynomial in ϕ are in some sense redundant. All of the physical information is already contained in the expansion:

$$V(\phi) = V_0 + V_1\phi + ... + V_{p-1}\phi^{p-1}, \tag{8.66}$$

so provided we only evaluate on $X_{\text{spacetime}}$, without loss of generality we can simply work solely in terms of this finite set of coefficients rather than the infinite set which is customary in effective field theory. Similar considerations hold for the kinetic term and any higher derivative term of fixed degree.[33] One consequence of this is that there is indeed some level of redundancy in higher order coefficients. This is in line with expectations on the constraints of a low energy effective field theory indicated in various Swampland conjectures (see e.g., [3, 118]). One can, of course, always include such higher degree terms, and this is useful in coming up with a better approximation scheme. It is, however, strictly speaking redundant information if we focus on just the structure of the possible values the effective potential can obtain.

On the other hand, the effective potential also makes implicit reference to an infinite series of coefficients V_i. In the broader setting of the BIG and big Hilbert spaces $\mathcal{H}_{\mathrm{BIG}}$ and $\mathcal{H}_{\mathrm{big}}$, we thus see an important physical consequence: Even though the potential can only ever attain a finite set of values, there is a strictly infinite set of possibilities for the power series in the effective potential. That being said, it remains unclear to us that an observer could ever access the full structure of these coefficients. The reason for this is that in any experiment, a measurement would need to report a value of the effective potential at specified points in spacetime.

Higher derivative corrections, however, do appear to be sensitive to arbitrarily high degree terms in the path integral. As we have already alluded to previously, this is one of the distinctions between other approaches to discretization, such as those associated with lattice approximations to a quantum field theory. To see why, recall that the rth Hasse derivative of a monomial u^n is given by:

$$\mathcal{D}^{(r)}u^m = \frac{m!}{r!(m-r)!}u^{m-r}. \tag{8.67}$$

when $0 \leq r \leq m$, and otherwise vanishes. As an extreme example, consider taking $n = p$ and $r = p$. In this case, we have $\mathcal{D}^{(p)}u^p = 1$. So, if we have a polynomial such as:

$$\phi(u) = \phi_0 + \phi_1 u^1 + \phi_2 u^2 + ... + \phi_p u^p + ..., \tag{8.68}$$

then the p^{th} Hasse derivative evaluated at $u = 0$ will return the coefficient ϕ_p. In a general effective action where we include arbitrary numbers of derivatives, we thus see that we cannot simply truncate to a finite number of coefficients. Of course, if we specialize our action so that it has only a finite number of derivatives, then we can again truncate, but from the general reasoning of effective field theory, there is no need to do so. In fact, we have also seen some crude analogs of unitarity in this setting. In the standard characteristic zero setting, such higher derivative interaction terms are necessary to ensure that the effective action remains unitary. This suggests that in our setting as well, one ought not to simply ignore such contributions. It would be interesting to explore the structure of such contributions

[33] However, we should also note that in a general effective action, there is an infinite sequence of higher derivative terms. These terms probe the structure of the higher degree terms of a given field configuration, and distinguishes our approach from the standard lattice approximation. We comment on this further on in this section.

and their impact on the structure of correlation functions.

8.4 Winding Modes and Double Fields

One of the items observed previously is that the structure of mode expansions in characteristic p has a somewhat different flavor than its characteristic zero analogs. In particular, we observe that in terms of the evaluation map on a finite field, there is a sense in which the actual value of the field has a large amount of redundancy, as evidenced by the appearance in ϕ_m^α of the momentum modes $m = 1, ..., q-1$ and the additional mode structure captured by $\alpha \in \mathbb{Z}$.

To further explore this redundancy in mode expansions, we now study the structure of winding modes in some simple examples. So, whereas in previous sections we considered maps from the punctured disk to the affine line, here we will instead focus on some more "topological aspects" of possible morphisms, as accounted for by $\pi_1^{\text{ét}}(X, x)$, the étale fundamental group of a scheme X at a geometric point x. See Appendix J for a brief discussion of the étale fundamental group.

To make things concrete, let us briefly revisit our general discussion of mode expansions. We have characterized a field as locally described by a power series expansion:[34]

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n u^n \in \mathbb{F}_q[[u, u^{-1}]] \tag{8.69}$$

Now, as we have already remarked, owing to the q^{th} Frobenius map $u \mapsto u^q$ leaving fixed all points of $\mathbb{F}_q$, and since we will eventually evaluate on such points, it is helpful to perform a further decomposition as:

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n u^n = \sum_{w \in \mathbb{Z}} \sum_{m=0}^{q-1} \phi_{m,w} u^{m+wq}. \tag{8.70}$$

We remark that this is quite similar to the previously discussed mode expansion, but that here the mode constraint is specified $\bmod\, q$ rather than $\bmod\, q-1$. We have also written w to indicate that we want to view this as a "winding number" and m to indicate that we are also dealing with a discretized momentum.

To explain the sense in which we are dealing with a winding mode, we now take the evaluation space to be points of the projective line $\mathbb{P}^1(\overline{\mathbb{F}}_q)$. We observe that in this case, the map $x \mapsto x^q$ leaves invariant the subscheme $\mathbb{P}^1(\mathbb{F}_q)$. In particular, we are now free to consider morphisms of the form:

$$\phi : \mathbb{P}^1(\overline{\mathbb{F}}_q) \to \mathbb{P}^1(\mathbb{F}_q), \tag{8.71}$$

[34] Here we do not demand that the expansion truncates at some degree. Of course, if we wish to evaluate the action we will ultimately need to truncate these expressions as in previous sections.

which we can view as factoring through some projective system of morphisms $\mathbb{P}^1(\mathbb{F}_{q_i}) \to \mathbb{P}^1(\mathbb{F}_q)$. Now, the point for us is that this is precisely the setup where we can fruitfully discuss winding maps, as captured by $\pi_1^{\text{ét}}(\mathbb{P}^1(\mathbb{F}_q), x) \simeq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}}$, where the profinite completion of the integers is just generated by the Frobenius map $u \mapsto u^q$, so in this sense, it really is appropriate to view the mode numbers w of line (8.70) as winding modes.

Now, in the context of string theory, the appearance of winding modes of course motivates a further question as to whether there is a sense in which momentum and winding modes can be interchanged on the target space. Here, we can already see one difficulty because whereas the set of winding numbers is formally infinite, the number of possible momenta is a finite set. Nevertheless, in the spirit of double field theory (see e.g. [119–121] and [122] for a review) we can consider a related mode expansion:

$$\Phi = \sum_{m,w} \Phi_{m,w} u^m v^w \in \mathbb{F}_q[[u, v, u^{-1}, v^{-1}]]. \tag{8.72}$$

We can then consider the restriction $v = u^q$, or alternatively the restriction $u = v^q$. Performing such a restriction, we that our expansion collapses back to either the presentation in terms of an expansion with local coordinate u or the T-dual coordinate v. In this situation, the roles of momentum and winding are clearly interchanged. Indeed, we can also consider a Lagrangian on the enlarged space, as given by:

$$L = \kappa \left((\partial_u \Phi)^2 - (\partial_v \Phi)^2 \right) + \text{Frobenius Conjugates}, \tag{8.73}$$

and we observe that there is an $SO(1,1)$ rotation amongst the local u and v coordinates. At the moment, this choice of relative sign seems like more of a natural possible choice rather than anything which is "forced" by consistency. Nevertheless, we can now see that maps from the worldsheet $\mathbb{P}^1(\overline{\mathbb{F}}_q)$ to the target space $\mathbb{P}^1(\mathbb{F}_q)$ do have a semblance of T-duality.

9 Green's Function on the Cylinder

In the previous section we encountered some examples of mode expansions, and evaluated some correlation functions. In the context of a standard characteristic zero field theory, a perhaps more natural starting point would have been to first analyze the *classical* equations of motion, and then expand in small fluctuations to evaluate corresponding correlation functions. This is more difficult in the characteristic p setting in part because many seemingly simple differential equations no longer make sense in this generalized setting. To illustrate the difficulties, consider the differential equation on the affine line $\mathbb{A}^1$:

$$\partial_v \phi(v) = \lambda \phi(v), \tag{9.1}$$

where we fix a choice of field K and take $\phi(v) \in K[[v, v^{-1}]]$ and take $\lambda \in K$. For $K = \mathbb{R}$ or $\mathbb{C}$ (as well as a p-adic field) we have the exponential function power series:

$$\exp(v) = \sum_{n \geq 0} \frac{v^n}{n!}, \tag{9.2}$$

and then the solution would be given by taking $\phi(v) = \exp(\lambda v)$. In the case of a characteristic p field, this will not work, because $1/n!$ makes no sense for $n \geq p$. On the other hand, in our discussion of mode expansions, we saw that on the cylinder, i.e., the punctured affine line $\mathbb{A}^{\times}$, it is natural to instead consider the differential operator:

$$D_u = u \frac{\partial}{\partial u}, \tag{9.3}$$

with u a local coordinate on $\mathbb{A}^{\times}$. In the case of $K = \mathbb{R}$ or $\mathbb{C}$, we can relate the local coordinate u with v via: $\exp(v) = u$. In particular, this provides us with a way to possibly make sense of $\exp(\lambda v)$ by instead working with u^{λ}. We can make sense of such an expression over $\mathbb{F}_p$ if we make the further assumption that $\lambda \in \mathbb{F}_p$. This follows because for any λ, we can consider $\widetilde{\lambda}$ a lift to $\mathbb{Z}$ such that the $\bmod p$ reduction of $\widetilde{\lambda}$ coincides with λ. So, we are free to write:

$$\widetilde{\lambda} = r + mp, \tag{9.4}$$

where $r \in \{0, ..., p-1\}$ and $m \in \mathbb{Z}$. Now, at this point we see that our solution to the differential equation takes the form:

$$\phi(u) = u^{r+mp}, \tag{9.5}$$

which solves the equation:

$$D_u \phi(u) = \lambda \phi(u). \tag{9.6}$$

So, we get an infinite number of formal solutions, namely we can write:

$$\phi(u) = u^r \sum_m \phi_m u^{mp}, \tag{9.7}$$

in the obvious notation. Note also that the same solution also works for the differential euqation:[35]

$$D_u^2 \phi(u) = \lambda^2 \phi(u). \tag{9.8}$$

Consider next the case where $\lambda \in \mathbb{F}_q$. Here we face the unpleasant feature that there will in general exist no lift of λ to an integer. As such, the interpretation of "u^λ" is unclear, at least to us. A similar workaround to what we already used is to consider introducing another variable "'$U = u^\lambda$", and with it a corresponding differential operator:

$$D_U = U \frac{\partial}{\partial U} = \frac{1}{\lambda} u \frac{\partial}{\partial u} = \frac{1}{\lambda} D_u. \tag{9.9}$$

In this case, we can work with expressions built from U rather than u, and in this case we observe that equation (9.8) instead becomes:

$$D_U^2 \phi(U) = \phi(U), \tag{9.10}$$

so at least formally, it suffices to restrict to the case of $\lambda \in \mathbb{F}_p$.

With this in mind, we now attempt to solve the classical source problem in characteristic p. More precisely, introduce $J(u) \in \mathbb{F}_p[[u, u^{-1}]]$. Our aim will be to solve the equation:

$$D_u^2 \phi(u) = J(u). \tag{9.11}$$

To begin, introduce explicit power series expressions:

$$\phi(u) = \sum_m \phi_m u^m \quad \text{and} \quad J(u) = \sum_m J_m u^m. \tag{9.12}$$

Then, we have, mode by mode:

$$m^2 \phi_m = J_m. \tag{9.13}$$

So, for $m \neq 0 \bmod p$, we can invert to find an explicit zero mode:

$$\phi_m = \frac{1}{m^2} J_m \quad \text{for} \quad m \neq 0 \bmod p. \tag{9.14}$$

In the case of $m = 0 \bmod p$, we cannot find a solution unless we assume $J_m = 0$ for such modes.

[35] One might refer to this as a harmonic oscillator equation since the notion of λ^2 positive or negative is meaningless in characteristic p. Observe, however, the existence / absence of a square root in $\mathbb{F}_p$ for $-\lambda^2$ depends on our choice of p.

10 Symmetries and Currents

Our discussion in the previous sections has laid out a general prescription for defining physics over characteristic p geometries. One aspect of our construction involves the analysis of correlation functions, and in principle, we can now proceed to evaluate examples of such quantities, much as we already did in section 8. Along these lines, it is well-appreciated that symmetries of a quantum system can provide important constraints on the structure of correlation functions. Our aim in this section will be to carry out a similar analysis in the characteristic p setting.

We shall refer to a "symmetry of the action" as any transformation either of the source X or target Y which leaves the character $\exp(iS/\hbar)$ invariant. In the quantum setting, it can often happen that a symmetry may nevertheless fail to leave the associated partition function invariant (as obtained from summing over all possible field configurations), and in this case, we refer to the symmetry as being "anomalous". We shall for the most part focus on an essentially classical analysis in the sense that our discussion focuses on invariance of $\exp(iS/\hbar)$.

Recall that in the standard characteristic zero setting, we can associate conserved currents with "continuous" symmetries. The main idea is to first fix a group action on the source X as well as the fields, and then peform a transformation which we label as $\phi \mapsto \phi^g$ in the obvious notation. In many cases of interest, this can be realized in terms of a linear transformation on a basis of fields ϕ^i, for example we can write $\phi^i \mapsto R^i_j \phi^j$. Indeed, we have already explained how gauge fields can be introduced in this setting, so it might indeed seem roundabout to now backtrack to discuss currents (which in standard treatments of field theory are often introduced first). One reason to defer our treatment to later is that the proper interpretation of symmetry currents requires some additional elements such as explicit mode expansions. Another reason has to do with the fact that in the characteristic p setting, we will also be able to associate a conserved current to discrete symmetries. At some level, this is not all that surprising, since our action principle automatically discretizes all physical quantities anyway. On the other hand, it allows us to provide a rather uniform treatment of different kinds of symmetries and gauge fields, some of which are a bit more cumbersome to construct in the continuum field theory setting. Let us also mention that the notion of assigning a conserved current to a discrete symmetries has recently been developed in the context of lattice quantum field theories [123].[36]

To begin, then, let us formalize the notion of a current in our setting. For ease of exposition, we work over $\mathbb{F}_q$ and consider a physical field $\phi : X \dashrightarrow Y$, and assume the existence of automorphisms $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ which respectively act on the source and target. Our goal will be to construct the corresponding current associated with a symmetry of the source or target. Again, rather than proceed in full generality, we proceed by way of

[36] Indeed, some of the considerations presented here were inspired by the discussion found in [123], as well as from discussion with the author D. Radičević.

example, which will hopefully suffice in filling in the sense in which we can indeed define a conserved current in this setting.

Our plan in the remainder of this section will be to construct some examples of currents for the target space and the source.

10.1 Target Space Examples

In this section we construct a current associated with a discrete symmetry of the target Y. We first consider the case of a $\mathbb{Z}/2\mathbb{Z}$ symmetry, and then proceed to a more elaborate class of examples based on $SL(2, \mathbb{F}_p)$.

We first illustrate how to construct a conserved associated with a discrete $\mathbb{Z}/2\mathbb{Z}$ symmetry of the target Y. To be precise, suppose we work over the punctured affine line, and consider physical fields $\phi(u) \in \mathbb{F}_q[u, u^{-1}]$. We fix our Lagrangian to be:

$$L[\phi] = \alpha D_u \phi D_u \phi - \beta \phi^2, \tag{10.1}$$

where as before, $D_u = u\partial_u$, and we have suppressed the appearance of a pairing $\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p$ to ensure that all configurations of $L[\phi]$ evaluate to $\mathbb{F}_p$ valued quantities. We observe that this action is invariant under the transformation $\phi(u) \mapsto -\phi(u)$. This action defines a "constant sheaf" in the sense that for each point in the source X, we take the same value of the group $\mathbb{Z}/2\mathbb{Z}$. To follow the Noether procedure, we now consider promoting this symmetry transformation to a local one, i.e., we consider the transformation $\phi(u) \mapsto g(u)\phi(u)$. Working to first order in the derivatives of $g(u)$, we can then hope to extract a conserved current, much as we would in the continuum setting.

So, the main condition we require is that no matter what point in $X = \mathbb{A}^\times$ we take, we require that $g(u)$ evaluates to 1 or -1. Moreover, we need to be able to find a set of spanning $g(u)$ in the sense that for each point in X, we can find a $g(u)$ which would evaluate to either 1 or -1 there (i.e., it is an arbitrary symmetry transformation). In the continuum setting, this cannot really be accomplished with a smooth function because large fluctuations from 1 to -1 necessarily create large discontinuities. It can be accommodated in the lattice setting (as in [123]), but again, the smooth limit is unclear.

The characteristic p setting is "simpler" in this regard. To illustrate, suppose we consider first the $q - 1$ non-zero points of $\mathbb{F}_q$, i.e., the point set of $\mathbb{A}^\times$. Let us label these points as $x_1, ..., x_{q-1}$. Now, we would like to find a $g(u)$ which generates a preferred set of signs $y_1, ..., y_{q-1}$, where each y_i is either 1 or -1 (i.e., bits). To find a $g(u)$ which accomplishes this task, we use the method of Lagrange interpolation. First, introduce the Lagrange polynomials:

$$l_j(u) = \prod_{k \neq j} \frac{u - x_k}{x_j - x_k} \in \mathbb{F}_q[u], \tag{10.2}$$

which satisfies $l_j(x_i) = \delta_{ij}$, the Kronecker delta. Then, the desired $g(u)$ is obtained from:

$$g(u) = \sum_j y_j l_j(x). \tag{10.3}$$

If we restrict to degree $q-1$ polynomials, this solution is unique. If we allow the degree to be larger, then we can of course consider more general $g(u)$'s which also evaluate to 1 or -1, and we can use these to build up local sections of the associated bundle. For example, we could consider instead $g(u^q)$, which would evaluate pointwise to the same values as $g(u)$, although it clearly has different derivatives (the case $g(u^q)$ has vanishing derivatives). We can also allow negative powers of u, since we can also take $g(u) \in \mathbb{F}_q[u, u^{-1}]$. For example, for $g(u)$ defined as in equation (10.3), the inverse $g(u)^{-1}$ makes sense for all points on the punctured affine line $\mathbb{A}^\times$.

With this in place, let us now determine the Noether current associated with this symmetry transformation. To obtain the associated current, we introduce a background gauge field and consider the Lagrangian:

$$L[A, \phi] = \alpha \mathcal{D}_u \phi \mathcal{D}_u \phi - \beta \phi^2, \tag{10.4}$$

for $\alpha, \beta \in \mathbb{F}_p$, in which the covariant derivative is given by:

$$\mathcal{D}_u \phi = D_u \phi + A_u \phi, \tag{10.5}$$

and where we have introduced a non-dynamical gauge field A_u. The $\mathbb{Z}/2\mathbb{Z}$ gauge transformations are then:

$$\phi(u) \mapsto g(u)\phi(u) \tag{10.6}$$

$$A_u \mapsto A_u + g(u) D_u g(u)^{-1} \tag{10.7}$$

$$\mathcal{D}_u \phi \mapsto g \mathcal{D}_u \phi, \tag{10.8}$$

where we require $g(u)$ to evaluate to 1 or -1 pointwise on the source X. Note that in the gauge field transformation, we have taken $g(u) D_u g(u)^{-1}$ rather than $g(u) D_u g(u)$. At the level of evaluating on point sets, the two choices are equivalent, but if one wishes to generalize to other discrete symmetries, it seems more appropriate to use the former rather than the latter.

From this, we conclude that $L[A, \phi]$ is indeed gauge invariant. To extract the associated current with this symmetry transformation, we now demand stationarity of the action under variations of A. Shifting $A \mapsto A + \delta A$, with δA a local one-form on X, we have:

$$S[A + \delta A, \phi] - S[A, \phi] = \sum_{t \in \mathbb{A}^\times} L[A + \delta A, \phi] - L[A, \phi] \tag{10.9}$$

$$= \alpha \sum_{x \in \mathbb{A}^\times} 2\delta A \phi D_u \phi + 2 (\delta A)^2 \phi^2. \tag{10.10}$$

Treating δA as an infinitesimal variation which formally satisfies $(\delta A)^2 = 0$, we observe that there is indeed a gauge field / current coupling, and the current is:

$$J_u = 2\alpha\phi D_u \phi. \tag{10.11}$$

Clearly, the entire derivation can be extended to the higher-dimensional setting by a suitable replacement of D_u.

In what sense is this current conserved? In the characteristic zero setting, the conservation equation for currents of continuous symmetries can be directly verified by imposing the on-shell equations of motion. In the present setting, the significance of imposing an equation such as $\alpha D_u^2 \phi + \beta\phi = 0$ is less clear. Indeed, if we restrict to physical field configurations which obey this condition, we observe that the resulting expression for $D_u J_u$ would be:

$$D_u J_u = 2\alpha \left(D_u \phi D_u \phi + \phi D_u^2 \phi \right) \tag{10.12}$$

$$= 2\alpha \left(D_u \phi D_u \phi - \frac{\beta}{\alpha} \phi^2 \right) \tag{10.13}$$

$$= 2L[\phi], \tag{10.14}$$

which in general is not zero. So, even though we can gauge a symmetry and observe that there is a standard background gauge field / current coupling, the notion of a conserved current is less apparent.

Though our discussion has primarily focused on the case of the punctured affine line, it should be clear that this notion generalizes in a straightforward way both to more general choices of characteristic p curves (with punctures), as well as higher-dimensional geometries. Let us briefly explain the sense in which our construction generalizes to other algebraic curves. In this setting, it is more appropriate to view the $g(u)$ as local sections of the sheaf of non-vanishing functions, denoted as $\mathbb{G}_m$. We remark that $\mathbb{G}_m$ is just the punctured affine line, viewed as a group under multiplication. We can also speak of μ_n, the sheaf of nth roots of unity, where it is simplest to assume that $\gcd(p, n) = 1$. In this case, one can establish that the ℓ-adic cohomology groups with $\mathbb{Z}/n\mathbb{Z}$ coefficients (identified with the sheaf of of nth roots of unity) on a genus g algebraic curve X satisfy:

$$\mathrm{rk} H^0_{\text{Ét}}(X, \mathbb{Z}/n\mathbb{Z}) = 1 \tag{10.15}$$

$$\mathrm{rk} H^1_{\text{Ét}}(X, \mathbb{Z}/n\mathbb{Z}) = 2g \tag{10.16}$$

$$\mathrm{rk} H^2_{\text{Ét}}(X, \mathbb{Z}/n\mathbb{Z}) = 1 \tag{10.17}$$

$$\mathrm{rk} H^i_{\text{Ét}}(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad i > 2, \tag{10.18}$$

namely it is the same as for a complex curve. For us, the point is that we can speak of $\mathbb{Z}/2\mathbb{Z}$

bundles, and the associated gauge theory which comes with it.

Nothing stops us from generalizing this to more general choices of discrete symmetries and currents. To give some "non-standard" examples, we can also consider the symmetry group $SL(2, \mathbb{F}_p)$, i.e., 2×2 matrices with entries in $\mathbb{F}_p$ with determinant $1 \in \mathbb{F}_p$. To build a corresponding theory which enjoys this symmetry, we introduce two doublets of physical fields ϕ^i and χ^i for $i = 1, 2$ which transform via matrix multiplication:

$$\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \tag{10.19}$$

$$\begin{bmatrix} \chi^1 \\ \chi^2 \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \chi^1 \\ \chi^2 \end{bmatrix}. \tag{10.20}$$

Then, we can introduce the kinetic term proportional to:

$$\varepsilon_{ij} D_u \phi^i D_u \chi^j = D_u \phi^1 D_u \chi^2 - D_u \phi^2 D_u \chi^1. \tag{10.21}$$

Observe that under a constant $SL(2, \mathbb{F}_p)$ transformation, we have:

$$\varepsilon_{ij} \partial \phi^i \partial \chi^j \mapsto \varepsilon_{ij} M^i_{i'} M^j_{j'} D_u \phi^{i'} D_u \chi^{j'} = \varepsilon_{ij} D_u \phi^i D_u \chi^j. \tag{10.22}$$

We would not follow this procedure in characteristic zero, because it has the unpleasant feature of creating "wrong sign kinetic terms" (i.e., ghosts) for propagating degrees of freedom. Indeed, we can introduce the explicit basis of fields:

$$\sigma_{\pm} = \phi^1 \pm \chi^2, \quad \pi_{\pm} = \chi^1 \pm \phi^2, \tag{10.23}$$

which diagonalizes the kinetic term at the expense of obscuring $SL(2, \mathbb{F}_p)$ invariance. Up to a constant of proportionality, we have:

$$\varepsilon_{ij} D_u \phi^i D_u \chi^j = D_u \sigma_+ D_u \sigma_+ + D_u \pi_+ D_u \pi_+ - D_u \sigma_- D_u \sigma_- - D_u \pi_- D_u \pi_-. \tag{10.24}$$

To introduce a corresponding gauge symmetry, we consider 2×2 matrices of the form:

$$g(u) = \begin{bmatrix} a(u) & b(u) \\ c(u) & d(u) \end{bmatrix}, \tag{10.25}$$

where $a(u), b(u), c(u), d(u) \in \mathbb{F}_p[u, u^{-1}]$ such that $ad - bc = 1$. The gauging procedure then proceeds much as one would expect, namely we introduce a covariant derivative:

$$\mathcal{D}_u \phi = D_u \phi + A_u \phi, \tag{10.26}$$

so that under a gauge transformation we have:

$$\phi \mapsto g(u)\phi \tag{10.27}$$

$$A_u \mapsto g(u)A_u g(u)^{-1} + g(u)D_u g(u)^{-1} \tag{10.28}$$

$$\mathcal{D}_u\phi \mapsto g(u)\mathcal{D}_u\phi, \tag{10.29}$$

and from this, we get the kinetic term:

$$\mathcal{L}_{\text{kin}} = \varepsilon_{ij} \left(\mathcal{D}_u\phi\right)^i \left(\mathcal{D}_u\chi\right)^j, \tag{10.30}$$

where we have set $\alpha = 1$ for simplicity. The current then follows from considering the linearized coupling to the gauge field:

$$\varepsilon_{ji} \left(A_u\right)^j_k \phi^k D_u\chi^i + \varepsilon_{ij} D_u\phi^i \left(A_u\right)^j_k \chi^k, \tag{10.31}$$

i.e., we have:

$$\left(J_u\right)^k_j = \varepsilon_{ji}\phi^k \left(D_u\chi^i\right) + \varepsilon_{ij} \left(D_u\phi^i\right) \chi^k \tag{10.32}$$

$$= \phi^k \left(D_u\chi_j\right) - \left(D_u\phi_j\right) \chi^k. \tag{10.33}$$

We remark in passing that the groups $SL(2, \mathbb{F}_p)$ also show up in surprising ways in the characteristic zero setting. For example, we have, for the binary tetrahedral and icosahedral groups:[37]

$$2T = SL(2, \mathbb{F}_3) \tag{10.34}$$

$$2I = SL(2, \mathbb{F}_5). \tag{10.35}$$

10.2 Spacetime Symmetry Examples

Let us now turn to currents involving the source or "spacetime". We view such symmetries as automorphisms of the spacetime X, which is the closest analog to a "diffeomorphism" we can entertain in the characteristic p setting. We would like to construct a corresponding Noether current associated with such symmetries.

Now, in the characteristic zero setting, the canonical example of a spacetime symmetry current is just the stress energy tensor of the theory. We obtain this by starting with the action for matter fields and varying with respect to the background metric:

$$T_{ab} = -\frac{2}{\sqrt{\det h}} \frac{\delta S_{\text{matt}}}{\delta h^{ab}}. \tag{10.36}$$

[37] The binary octahedral group is instead given by $CSU(2, \mathbb{F}_3)$, the group of conformal special unitary matrices acting on $\mathbb{F}_3$. For additional examples of such groups, see [124].

Returning to the characteristic p setting, we already face some difficulties with this interpretation, because we need to have a notion of the "metric" as well as quantities such as $\det h$. As we already explained in subsection 4.5, there is still a notion of $\sqrt{\det h} \in \Omega^m(X, \mathcal{K}_X)$ which implicitly varies as we change the choice of symmetric bilinear form $h : T^*X \times T^*X \to K$. So, at least at the level of varying our action, we can still make sense of the variational problem specified by equation (10.36). The main subtlety we face, as in our previous discussions, is that now, we cannot simply drop "surface terms," since we do not possess the same notion of Stokes' theorem in the characteristic p setting. Nevertheless, at the level of a variational principle, we can still speak of the corresponding current associated with diffeomorphisms.

Now, in addition to these "continuous" transformations, this procedure also automatically includes symmetries which one might view as "discrete transformations". For example, in QED in four dimensions, the theory is also invariant under parity, and time reversal transformations, namely spacetime symmetries. In the present setting, there is little distinction; all of the automorphisms of X are on the same footing, and the corresponding Noether current equally applies to the "continuous" and discrete automorphisms of X.

11 Topological Actions

There is a sense in which formulating a physical theory over a characteristic p geometry is necessarily topological, simply because the number of points in both the spacetime and the target space are already discretized. Of course, one of the important features of such geometries is that whereas notions such as the Zariski topology are quite coarse, more refined étale topologies provide a closer link to the expected realm of characteristic zero geometry. In this section we discuss in greater detail the sense in which we can formulate topological actions with respect to an étale topology.

For starters, let us explain the sense in which the actions we have been discussing so far need not be topological. By way of example, consider the free scalar on the punctured affine line $\mathbb{A}^{\times}$ over $\mathbb{F}_p$, with Lagrangian:

$$L[\phi] = \alpha D_u \phi D_u \phi - \beta \phi^2. \tag{11.1}$$

Now, in the path integral, we have weighted each field configure by a factor of $\exp(2\pi i S/p)$. This sort of weighting by a p^{th} root of unity is of course quite reminiscent of various topological actions one encounters in various Chern-Simons theory and BF theories as well as more general Dijkgraaf-Witten theories [125]. As an example, consider a three-manifold M_3 and an abelian (spin) Chern-Simons theory with two gauge fields a and A with action:

$$S_{\text{CS}}[A, a] = \frac{1}{4\pi} \int\limits_{M_3} 2A \wedge da - Na \wedge da. \tag{11.2}$$

The equations of motion for a relate the curvatures f and F respectively for a and A as $f = F/N$, so making use of this equation of motion takes us to an "improperly quantized" Chern-Simons action for just A:

$$\text{``}S_{\text{imp}}[A] = \frac{1}{4\pi N} \int\limits_{M_3} A \wedge dA\text{''}. \tag{11.3}$$

The factor of $1/N$ in the action is quite reminiscent of our proposal to take $\hbar = p/2\pi$. Of course, the proper action to use is really $S_{CS}[A, a]$ rather than $S_{\text{imp}}[A, a]$, in part because the former is invariant under large gauge transformations whereas the latter is not.

Another important difference is that by design, the Chern-Simons action does not depend on the choice of a metric on the three-manifold M_3. This is to be contrasted with the action implicitly defined in equation (11.1). Although there is no notion of "metric" per se, we have already mentioned that there is an implicit dependence on a symmetric bilinear form $h : T^*X \otimes T^*X \to \mathbb{F}_p$, so it violates the spirit of building a topological action. Another way of saying the same thing is that although our action is of course trivial with respect to the Zariski topology, the pairing h depends on a choice of étale covering $X_i \to X$.

In [89–92] a proposal is given for defining an arithmetic Chern-Simons theory, and this is further generalized in [126] to various Dijkgraaf-Witten topological theories [125].[38] Intriguingly, the main player in defining an arithmetic "classical" Chern-Simons invariant is the appearance of a $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ valued functional, which is also rather close to the considerations we have been discussing. On the other hand, we will be formulating our action over a threefold (in the case of Chern-Simons theory) rather than appealing to any analogy between points in $\operatorname{Spec}\mathbb{Z}$ and knots in real three-manifolds. So, rather than directly make contact with this interesting proposal, we shall instead attempt to directly construct the characteristic p version of our action.

To proceed, we now discuss an example of a topological action and its generalization to the characteristic p setting. We primarily focus on abelian Chern-Simons theory on a three-manifold:

$$S_{\text{CS}}[A] = \frac{k}{4\pi}\int A \wedge dA = \frac{k}{4\pi}\int d^3x\ \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{11.4}$$

Here, A is a one-form connection for an abelian gauge group. We have already argued that we can still speak of the gauge connection on a characteristic p space, so the main obstruction we face appears to be the existence of a suitable notion of a three-index tensor $\varepsilon^{\mu\nu\rho}$ for the Chern-Simons action.

It is already instructive to ask about what happens if we take the above characteristic zero actions and extend the ground field from $\mathbb{R}$ to $\mathbb{C}$. There is a simple way to extend all of our actions simply by treating A as a $(0,1)$-form. Then, assuming the existence of a suitable three-index object amounts to to requiring a non-vanishing section of $H^0(X, \mathcal{K}_X)$ with $\mathcal{K}_X$ the canonical sheaf.[39] Said differently, if X is Calabi-Yau, namely it has trivial canonical sheaf, then we can specify a classical action. For a threefold we can then write:

$$S_{\text{hol-CS on } X_3} = \int_{X_3} \Omega_{(3,0)} \wedge A_{(0,1)} \wedge \overline{\partial} A_{(0,1)}. \tag{11.5}$$

There are two general issues we encounter in specifying such an action. First of all, the action is not even real valued. This is not much of an issue in the context of physical superstring computations because the holomorphic Chern-Simons action, for example, is mainly used to extract superpotential couplings (via the target space formulation of the topological B-model). Of course, we can produce a real action by adding the complex conjugate to any configuration in a trivial manner.

Another issue is that our action is not really gauge invariant, and is instead only defined modulo periods of the holomorphic three-form of the Calabi-Yau space. There is a simple workaround for this which is to extend the Calabi-Yau threefold to a Fano fourfold Y with a meromorphic $(4,0)$-form $\Omega_{(4,0)}$. The pole for this meromorphic form is a Calabi-Yau threefold

[38] For a different way in which number theoretic structures enter in the discussion of topological field theories, see for example [127] and references therein.

[39] This follows from Serre duality for a threefold X. We have $H^3(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{K}_X)^\vee$.

(as follows from adjunction), and we take this to be X, the original space of interest. We can then introduce a corresponding manifestly bulk gauge invariant action for the holmorphic Chern-Simons action:

$$S_{\text{FF on } Y_4} = \int\limits_{Y_4} \Omega_{(4,0)} \wedge \overline{\partial} A_{(0,1)} \wedge \overline{\partial} A_{(0,1)}, \tag{11.6}$$

which is now manifestly gauge invariant. The price we pay is that now we are making reference to a specific bulk space Y_4 to define our action, but this at least has the virtue of being well-defined.

We now turn to the characteristic p analog of this structure. Nothing stops us from introducing the analog of the Chern-Simons action, and again we face the same issues with the action being defined modulo periods of $\Omega \in H^3(X, \mathcal{O}_X) \simeq K$. To address this, we view X_3 as a divisor of an ambient Y_4, and instead work with respect to the topological action defined on Y_4. We can then restrict as necessary to the subspace cut out by the pole of the four-form on Y_4. From this perspective, we can introduce the action:

$$S_{\text{FF on } Y_4} = \sum_{y \in Y_4} \text{ev}_{u=y} \left(\Omega \wedge dA \wedge dA \right), \tag{11.7}$$

in the obvious notation. Much as in other contexts we already considered, we can produce an $\mathbb{F}_p$-valued action by first working over K a finite extension of $\mathbb{F}_p$ and then performing a trace over $K/\mathbb{F}_p$.

At some level, it would be more satisfactory if we could directly dispense with the appearance of a bounding space Y_4. For example, we can perform precisely the same analysis in the context of ordinary 3D Chern-Simons theory, where the presence of an improperly quantized level means that to truly define the theory we must view as a boundary term of a 4D topological theory with action proportional to $\theta F \wedge F$ with θ circle valued. There is a special decoupling which occurs when the level is properly quantized, and so it is natural to ask whether the characteristic p situation is actually more similar to working over $\mathbb{C}$, or closer to the real case.

So, let us simply write down a Chern-Simons action and check whether it is gauge invariant. Our proposed action on X a threefold is:

$$S_{\text{CS on } X_3} = k \sum_{x \in X} \text{ev}_{u=x} \left(\Omega \wedge A \wedge dA \right), \tag{11.8}$$

for abelian Chern-Simons, with the obvious generalization for non-abelian Chern-Simons theory. The basic issue we face is that under a large gauge transformation, the action will not necessarily return to itself. For abelian Chern-Simons theory, this is detected through the fundamental group of the underlying three-manifold, while for non-abelian Chern-Simons theory it is captured by the "winding number" as specified by the classification of maps $g : M_3 \to G$. We do have suitable notions of an étale fundamental group as well as higher etale homotopy groups (see e.g., Appendix E of reference [128]) so it is of course natural to ask whether we can make sense of possible issues with "winding field configurations" in our topological action. To keep things concrete, work over the ground field $K = \mathbb{F}_q$ and suppose our threefold is actually a product $\mathbb{A}^\times \times S$.

Our main interest will be in general gauge transformations of the form $A \mapsto A + g^{-1}dg$, where $g(u) \in K(u, u^{-1})$ is such that all evaluations $\mathrm{ev}_{u=x}(g(u))$ for $x \in \mathbb{A}^\times$ is invertible. As a warmup, consider $g(u) = u^l$ for arbitrary $l \in \mathbb{Z}^\times$. In this case, we have:

$$g^{-1}dg = lu^{-1}du. \tag{11.9}$$

Observe that for characteristic $p \neq 2$, we have (see equation (8.4)):

$$\sum_{x \in \mathbb{F}_q^\times} u^{-1} = 0. \tag{11.10}$$

So at least for this class of gauge transformations, the action is invariant. Note that this also includes the topologically non-trivial case involving "winding modes", i.e., $g(u) = u^p$ the Frobenius map (see section 8.4). Indeed, such winding modes are in one to one correspondence with elements of $\mathrm{Gal}(K/\mathbb{F}_p) \simeq \mathbb{Z}/d\mathbb{Z}$ with d the degree of the field extension in question, and the Frobenius map the generator of $\mathbb{Z}/d\mathbb{Z}$. More generally, consider any $g(u)$ an automorphism $\mathbb{A}^\times \to \mathbb{A}^\times$. We can perform a change of local coordinates, reducing to the case of a polynomial $g(u)$. From this, we conclude that our action is, in fact gauge invariant.

Similar considerations hold for other topological actions and their characteristic p counterparts. For starters, we can generalize to the case of a non-abelian Chern-Simons theory. In this case, the gauge transformations will produce terms of the form $\mathrm{Tr}(g^{-1}dg)^3$, which provides a "practical" definition of winding numbers for the corresponding maps $g : X_3 \to G$ with G the non-abelian gauge group. As an another case of interest, we can consider the four-dimensional BF theory on a four-manifold M_4 specified by the choice of a 0-connection A and a 1-connection B for an abelian gerbe:

$$S_{\mathrm{BF}} = \frac{k}{2\pi} \int B \wedge dA = \frac{k}{2\pi} \int d^4x \; \varepsilon^{\mu\nu\rho\rho} B_{\mu\nu} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right). \tag{11.11}$$

From this, it does appear that there is a natural characteristic p version of various topological actions, and that is actually somewhat more direct when compared with working over $\mathbb{C}$ (as opposed to $\mathbb{R}$).

Our approach so far has been rooted in building explicit actions. This is rather different from the mathematical approach to defining and studying topological field theories (TFTs) via the Atiyah-Segal axioms [129, 130], where one specifies a TFT as a symmetric monoidal functor $Z : \mathrm{Bord}_n^\xi \to (\mathrm{Vec}_{\mathbb{C}}, \otimes)$ for Bord_n^ξ bordism classes of n-manifolds (with $(n-1)$-dimensional boundaries) equipped with some choice of ξ-structure. This can be generalized (i.e., categorified) in various ways, but the main point is that we need a notion of specifying a partition function $Z(M)$ on a manifold M equipped with some structure such that we have a suitable notion of cutting and gluing, as captured by cobordisms.

Does this have any meaning in the characteristic p setting? The first question we need to address is whether we can even specify a suitable algebro-geometric definition of cobordism. In fact, for complex bordisms there appears to be a suitable generalization to algebraic bordism which applies for any characteristic zero field (for example the p-adics) [131–133]. In that setting, it is important that one works over characteristic zero to have Hironaka's theorem for resolving singularities. That being said, it would appear that nothing prevents us from working p-adically and then performing a suitable reduction mod p.[40] The fact that we have a candidate class of actions available, and thus corresponding partition functions, suggests that the main bottleneck is indeed simply coming up with an appropriate generalization of Bord_n^{ξ}. Perhaps the answer is to be found in [128].

[40]This is especially natural in the context of p-adic analytic spaces, a topic we briefly touch upon in section 20.

12 Physical Twistors and Amplitudes

In the previous sections we have proposed a general path integral formalism as well as a Hilbert space interpretation for physical systems on characteristic p geometries. Now, one of the subtle points we have already encountered is that the notion of "time ordering" in the characteristic p setting involves making some choices with regards to what we mean by time evolution in the first place. That being said, we have also seen that for the punctured affine line, we have a notion of "past and future," so it does appear to make sense to speak about scattering amplitudes and other related observables and we can use the standard link between scattering amplitudes and time ordered correlation functions to at least formally define these notions.

In this section we further analyze the extent to which we can expect such notions to extend to the characteristic p setting. To keep things concrete, we focus on the case of four-dimensional field theories with a natural conformal structure in the characteristic zero setting. In this case, we can fruitfully borrow many notions from twistor geometry to recast questions concerning the causal structure in the four-dimensional setting in terms of algebro-geometric structures in three complex dimensions. At this point, our line of approach ought to be clear: once we recast our questions in algebro-geometric terms, we can pass over to the arithmetic setting. In Appendix L we review some elements of twistors for real and complexified spacetimes, so we assume the main elements of this discussion in what follows. In what follows we primarily work over the algebraic closure $\overline{\mathbb{F}}_p$, but also consider the case where we restrict to $\mathbb{F}_p$. Our plan will be to first introduce a notion of physical twistors in characteristic p, and to then discuss solutions to wave equations in this setting, viewed as elements of bundle valued cohomology groups. Scattering amplitudes implicitly follow as functions which depend on these bundle valued cohomology groups.

12.1 Twistor Space

Our starting point will be conformally compactified Minkowski space $\overline{\mathbb{F}}_p\mathbb{M}^{\#}$ as specified by the quadric in $\overline{\mathbb{F}}_p\mathbb{P}^5$:[41]

$$\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}R^{\gamma\delta} = 0, \tag{12.1}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the four-index epsilon tensor and $R^{\alpha\beta} = -R^{\beta\alpha}$ with $\alpha, \beta = 1, 2, 3, 4$, namely the $R^{\alpha\beta}$ denote homogeneous coordinates of $\overline{\mathbb{F}}_p\mathbb{P}^5$. Raising and lowering of pairs of indices is accomplished via (we adopt the standard physics conventions which are acceptable at least when $p \neq 2$):

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta} = R_{\gamma\delta}. \tag{12.2}$$

[41] A word on notation. We have been indicated the choice of ground field for the affine line and other spaces by $\mathbb{A}^1(\overline{\mathbb{F}_p}$, but here we have put the choice of ground field "to the left". This is more to make comparison with the standard twistor theory literature.

We can formally solve the quadric equation $\varepsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta} R^{\gamma\delta} = 0$ by introducing two copies of projective twistor space $\mathbb{PT} \simeq \overline{\mathbb{F}}_p \mathbb{P}^3$ with homogeneous coordinates Z^α and W^β via:

$$R^{\alpha\beta} = Z^\alpha W^\beta - Z^\beta W^\alpha. \tag{12.3}$$

So, in the twistor perspective, a pair of twistor points specify a point in the spacetime. Since a point in twistor space can also be viewed as specifying a divisor, a pair of divisors can, via their intersection, specify a line, namely an $\overline{\mathbb{F}}_p \mathbb{P}^1$. So, each point in our spacetime specifies a line in twistor space.

Now, once we introduce a suitable infinity bitwistor $I_{\alpha\beta}$ we can indicate the region of $\overline{\mathbb{F}}_p \mathbb{M}^\#$ to delete via the equation:

$$I_{\alpha\beta} R^{\alpha\beta} = 0 \tag{12.4}$$

where for Minkowski space, we demand:

$$I_{\alpha\beta} I^{\alpha\beta} = 0. \tag{12.5}$$

The discussion is purely algebraic, and therefore parallels what can be done in real and complex space, so we can essentially appeal to the discussion given in Appendix L. In particular, we make the choice that

$$I_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{12.6}$$

in which case the line at infinity is specified by setting $R^{34} = 0$. We can now meaningfully split the homogeneous coordinates up as $Z^\alpha = (Z^1, Z^2, Z^3, Z^4) = (\omega^1, \omega^2, \pi_{1'}, \pi_{2'})$. In this case, there is a distinguished line, the "twistor at infinity" given by

$$\mathbb{P}^1_\infty = \{\pi_{1'} = \pi_{2'} = 0\}, \tag{12.7}$$

and we can refer to the space with this $\mathbb{P}^1_\infty$ deleted as $\mathbb{PT}'$. In practice, what we really mean by this is that we allow ourselves to consider various sections of bundles with poles along this $\mathbb{P}^1_\infty$.

Let us now turn to the characterization of the points which are not at infinity in Minkowski space so that $R^{34} \neq 0$. In this chart, it is helpful to introduce a 2×2 position matrix $x^{AA'}$ with entries:

$$x^{AA'} = \frac{1}{\widehat{i}} \begin{bmatrix} R^{14}/R^{34} & -R^{13}/R^{34} \\ R^{24}/R^{34} & -R^{23}/R^{34} \end{bmatrix}. \tag{12.8}$$

Here, we have introduced the number $\widehat{i}$ which satisfies $F(\widehat{i}) = -\widehat{i}$ with F Frobenius conjugation. In this patch, Minkowski space is represented as the paraboloid:

$$\frac{R^{12}}{R^{34}} + \det x = 0 \tag{12.9}$$

as follows from substitution into the quadric equation. In these variables, the incidence relation is:

$$\omega^{A} = \widehat{i} x^{AA'} \pi_{A'}. \tag{12.10}$$

The locus of points associated with $\mathbb{F}_p$ Minkowski space, are those points in $\mathbb{PT}'$ for which $F(x^{AA'}) = x^{AA'}$.

12.2 Zero Modes and Amplitudes

Now, one of the elegant applications of twistor methods is in the study of zero mode solutions for massless fields in Minkowski space. This approach exploits the conformal structure of such massless systems. As discussed for example in [134–137], we can speak of a state of helicity h as being captured by an element of the cohomology group $H^1(\mathbb{PT}', \mathcal{O}(2h-2))$. Here, we have been deliberately imprecise about a specific choice of a cohomology theory. Presumably, the physically sensible case is associated with rigid cohomology or some closely related variant (we discuss some possibilities later in section 13).

Now, in the context of scattering theory of massless particles, we are accustomed to specifying a state of null momentum in terms of an outer product:

$$P_{AA'} = \lambda_A \widetilde{\lambda}_{A'}, \tag{12.11}$$

where λ_A and $\widetilde{\lambda}_{A'}$ are two-component spinors. Now, these objects have opposite degree of homogeneity. In the spirit of reference [137], we can work directly in terms of functions on twistor space by "Fourier transforming" one of these helicity variables. The meaning of Fourier transform is unclear in characteristic p, but does make sense in the context of p-adic geometry. At any rate, nothing stops us from directly defining physical quantities on the projective twistor space $\mathbb{PT}'$, and once we do so we can basically borrow the analysis of reference [137] where we interpret scattering amplitudes on (momentum) twistor space in terms of the geometry of such twistorial objects. For example, quantities such as the leading order contribution to the color-stripped n-gluon MHV amplitude $(-,-,+,...,+)$ computed in reference [138] (we use standard spinor-helicity conventions):

$$\mathcal{M} \sim \frac{\left\langle \lambda^{(s)}, \lambda^{(t)} \right\rangle^4}{\left\langle \lambda^{(1)}, \lambda^{(2)} \right\rangle ... \left\langle \lambda^{(n)}, \lambda^{(1)} \right\rangle}, \tag{12.12}$$

where there is an implicit momentum conservation condition has been inserted, and particles s and t have -1 helicity while all others have helicity $+1$. Our main point is that such quantities still make sense as quantities defined over $\mathbb{PT}'$ with ground field $\overline{\mathbb{F}}_p$ as do more general scattering amplitudes.

A potential objection to this way of proceeding is that whereas the leading order tree level behavior of scattering amplitudes has a simple presentation in terms of meromorphic sections of bundles on twister space, loop corrections to this structure necessarily involve the appearance of transcendental functions of the momenta. Owing to this, one might ask whether any generalization is available which would also apply to such situations.

As a first comment along these lines, we remark that at least for unitary theories, one expects that the scattering amplitudes obtained are analytic functions of the Mandelstam parameters, aside from possible branch cut singularities. Expanding around any given region in complexified kinematic variables, we can then ask whether it makes sense to discuss such structures in the characteristic p setting. Indeed, one answer we can provide is that there is indeed a notion of formal Laurent series such as $\overline{\mathbb{F}}_p[[u, u^{-1}]]$, and we can restrict to the physical case where the degree of the inverse powers is bounded, as associated with $\overline{\mathbb{F}}_p((u))$. In fact, one can also generalize this to accommodate fractional powers; for example in the space $\overline{\mathbb{F}}_p\{\{u\}\}$ of Puiseux series we deal with fractional powers, which we can obtain from the direct limit on n of $\overline{F}_p((u^{1/n})$.

A somewhat unsatisfactory element of such an answer is that the amplitude is no longer really a "number" but instead a formal power series. This too can be rectified by performing a further lift to an expression defined over the p-adics. In this case, the main issue would seem to be that the radius of convergence for a p-adic expression can be rather different from its more standard counterpart defined over $\mathbb{C}$. For additional discussion of the p-adic exponential, see e.g., Appendix Q, and for additional discussion of the p-adic (poly)logarithm, see e.g., Appendix R. Nevertheless, the main point we wish to emphasize here is that insofar as the analysis of scattering amplitudes rests on objects "visible" within the scope of algebraic geometry, there is an implicit prescription available for recasting these structures over other choices of ground field.[42]

[42]For some additional discussion in the context of non-Archimedean geometry in mixed characteristic, as well as its potential relations to tame geometry, see section 22.

13 Fermionic Systems

Our discussion up to this point has focused on systems involving bosonic degrees of freedom. In this section we develop a parallel story for fermionic degrees of freedom. We shall make use of the main geometric elements for bosonic systems developed previously. For now, we again restrict to the special case where:

$$\hbar = \frac{p}{2\pi}, \tag{13.1}$$

with p a prime. For some earlier discussions of arithmetic with Grassmann algebras see e.g., [139, 140].

As before, we motivate our analysis by beginning with a quantum mechanical system with discretized observables. In this case, we consider a two state Hilbert space spanned by the states $|\uparrow\rangle$ and $|\downarrow\rangle$, and introduce fermionic operators $\widehat{b}$ and $\widehat{c}$ which satisfy the algebra:

$$\left\{\widehat{b}, \widehat{c}\right\} = 1, \quad \widehat{b}^2 = \widehat{c}^2 = 0. \tag{13.2}$$

These operators act on our states as follows:[43]

$$\widehat{b}\,|\downarrow\rangle = 0, \quad \widehat{b}\,|\uparrow\rangle = |\downarrow\rangle \tag{13.3}$$

$$\widehat{c}\,|\downarrow\rangle = |\uparrow\rangle\,, \quad \widehat{c}\,|\uparrow\rangle = 0. \tag{13.4}$$

As an example, we can consider the Hamiltonian operator:

$$\widehat{H} = m\widehat{c}\widehat{b}. \tag{13.5}$$

Time evolution of states is accomplished by acting with the unitary operator:

$$U(t) = \exp(-i\widehat{H}t/\hbar), \tag{13.6}$$

where we assume (as discussed previously) that we can only make measurements in a smallest time step $t \in \mathbb{Z}$. Even though the fermionic degrees of freedom are already discretized, one might ask whether there are any restrictions on the parameter m. Observe that the explicit form of our time evolution operator is:

$$U(t) = \exp\left(-\frac{2\pi i}{p} m\widehat{c}\widehat{b}t\right). \tag{13.7}$$

[43] By inspection, $\widehat{c} = \widehat{b}^\dagger$.

Acting on the two states, we have:

$$U(t)\left|\downarrow\right\rangle = \sum_{n\geq 0}\frac{1}{n!}\left(-\frac{2\pi i}{p}m\widehat{cbt}\right)^{n}\left|\downarrow\right\rangle = 0 \tag{13.8}$$

$$U(t)\left|\uparrow\right\rangle = \sum_{n\geq 0}\frac{1}{n!}\left(-\frac{2\pi i}{p}m\widehat{cbt}\right)^{n}\left|\uparrow\right\rangle = \exp\left(-\frac{2\pi i}{p}mt\right)\left|\uparrow\right\rangle. \tag{13.9}$$

Provided we restrict m to the integers, we see that the complex phase for $\left|\uparrow\right\rangle$ will eventually return after at most p time steps. With this motivation in mind, we can now proceed to develop the parallel formalism for fermionic path integrals. This is an entirely standard development in characteristic zero, and is covered in detail for example in [141] and Appendix A of [142].

What is not so standard is to understand the characteristic p version of fermionic systems. Here, we will aim to convey the main physical issues. The first issue we face is that we will need to supplement the finite field $\mathbb{F}_p$ by anti-commuting Grassmann variables. We define Grassmann variables χ_i by requiring that they anti-commute, i.e.:

$$\chi_i\chi_j = -\chi_j\chi_i. \tag{13.10}$$

The appropriate notion of an $\mathbb{F}_p$-valued Grassmann variable in this setting will be that the extension of the Frobenius endomorphism F to anticommuting variables leaves such variables fixed. With this in mind, we require that an $\mathbb{F}_p$-Grassmann variable χ satisfies:

$$F(\chi) = \chi, \tag{13.11}$$

which is the analog of Hermitian conjugation in characteristic zero. We also demand that for any bosonic $\phi \in \mathbb{F}_p$ that we have:

$$F(\phi\chi) = \phi\chi. \tag{13.12}$$

Now, given multiple $\mathbb{F}_p$ Grassmann variables, we would like to extend the action of the Frobenius automorphism to products of Grassmann variables.

A priori, there are two ways in which one might attempt to proceed. On the one hand, if we insist on keeping all coefficients valued in $\mathbb{F}_p$, we can consider an action which respects multiplicative order. On the other hand, we can allow the Frobenius map to switch the order of fermions:

$$F_{\text{option }1}(\chi\psi) = F(\chi)F(\psi) = \chi\psi, \tag{13.13}$$

$$F_{\text{option }2}(\chi\psi) = F(\psi)F(\chi) = \psi\chi. \tag{13.14}$$

In the physical setting, it is more natural to treat Grassmann fields as operators, so a conjugation operation would switch the order of multiplication. We therefore focus on "option 2." The case of "option 1" is of interest in its own right, however, and we discuss how to build supersymmetric actions with this choice in Appendix M. One can view the two procedures as related by analytic continuation, i.e., by multiplying some fields by appropriate "imaginary numbers."

Now, in characteristic zero, we are often interested in Hermitian operators built from products of such fermionic fields. The way we do this involves multiplication by factors of $i = \sqrt{-1}$, since complex conjugation reverse the order of multiplication on Grassman fields and $i^* = -i$. We need to introduce a suitable notion of "i" which flips sign under Frobenius conjugation. We already encountered this feature in our discussion of vector potentials in section 4.4, where we noted that $\sqrt{-1}$ sometimes will not accomplish this goal. For example, in $\mathbb{F}_5$, observe that $3^2 = -1$. Just as in section 4.4, we will instead seek out a root of the polynomial equation:

$$x^p = -x, \tag{13.15}$$

and we denote one such root by $\widehat{i}$. Observe that by design, we have, under Frobenius conjugation:

$$F(\widehat{i}) = \widehat{i}^p = -\widehat{i}. \tag{13.16}$$

Since $\widehat{i}$ is not invariant under Frobenius conjugation, it is not an element of $\mathbb{F}_p$. Note, however, that its square $\widehat{i}^2$ is invariant, and is therefore an element of $\mathbb{F}_p$. What we cannot assert, however, is that $\widehat{i}^2 = -1$. Indeed, $\widehat{i}$ is an element of $\mathbb{F}_q$ with $q = p^2$. A combination of $\mathbb{F}_p$-valued Grassmann numbers invariant under Frobenius conjugation can now be obtained through a product such as:

$$\widehat{i}\chi\psi. \tag{13.17}$$

Having set our conventions for Grassmann coordinates in characteristic p, we can now proceed to build fermionic actions. As a warmup, we first develop the 1D path integral. Introduce a formal parameter u and expand our fermionic fields via the power series:

$$\chi(u) = \sum_m \chi_m u^m \quad \text{and} \quad \psi(u) = \sum_m \psi_m u^m, \tag{13.18}$$

where each of the coefficients is an $\mathbb{F}_p$-Grassmann variable. Returning to our two state system, the Lagrangian will be viewed as a Grassmann even polynomial in the variable u, and the action is obtained through the evaluation map:

$$S = \sum_{x \in X} \mathrm{ev}_{u=x} \left(\widehat{i}\chi \partial_u \psi - \widehat{i} m \chi \psi \right), \tag{13.19}$$

namely we evaluate to an $\mathbb{F}_p$ valued Grassmann bilinear. Indeed, each term in the above sum is invariant under Frobenius conjugation, and should thus be viewed as $\mathbb{F}_p$ valued.

Evaluation of the path integral now proceeds just as in characteristic zero; We can perform Grassmann integrals by expanding the exponentials, and evaluate fermionic correlation functions in the standard way.

This generalizes to other spacetime dimensions. With conventions as in subsection 4, we introduce a polynomial ring $\mathbb{F}_q[u_1, ..., u_D]$ for our bosonic physical fields. We can supplement this by tensoring with a set of Grassmann coordinates. Along these lines, recall from equation (4.5) that a bosonic physical field was initially presented as a power series expansion:

$$\phi(u_1, ..., u_D) = \sum_{m_1, ..., m_D} \phi_{m_1 ... m_D} (u_1)^{m_1} ... (u_D)^{m_D}, \tag{13.20}$$

We can write a fermionic analog of this by expanding with Grassmann valued coefficients:

$$\chi(u_1, ..., u_D) = \sum_{m_1, ..., m_D} \chi_{m_1 ... m_D} (u_1)^{m_1} ... (u_D)^{m_D}, \tag{13.21}$$

where each coefficient $\chi_{m_1 ... m_D}$ is to be treated as a Grassmann coordinate. Now, in the bosonic case, the path integral instruction is to sum over all these choices of $\phi_{i_1 ... i_D}$. In the fermionic context, we perform a Grassmann integral. So, we can again construct Lagrangians and actions for our physical fields. The only difference now is that there will be some Grassmann dependence. The main condition we impose is that the coefficients of any expression in our action are again $\mathbb{F}_p$ valued. Again, this is the analog of a "reality condition" in the characteristic p context.

What sort of correlation functions should we consider computing in this context? As in the case of purely bosonic systems, we observed that operators which respect our reduction modulo p are the ones of interest. In the fermionic context, the standard expansion of Grassmann integrals might suggest that this is not possible. Of course, in quantum field theory we are accustomed to viewing operators constructed from composite fermions as bosonic objects. This in turn means that in this setting, the simplest class of operator correlation functions to consider are those which are built from such bosonic operators. An example of this sort is the time evolution operator of our two level system introduced in equation (13.6).

Proceeding along the same steps following for our bosonic field theory, we can extend all of these considerations to far more general spacetimes X and target spaces Y. In this more general setting, it is appropriate to replace our polynomials by expressions which are locally rational functions. This is acceptable provided we specify what happens at the singularities of the evaluation map.

Now, up to this point we have ignored the spin of our fermionic degrees of freedom. In characteristic zero, one can locally speak of a spinor bundle, and in suitable circumstances this extends to the global manifold. In more algebraic terms, we can introduce a sheaf of spinors $\mathcal{S}$ such that along each stalk $\mathcal{S}_x$, we have a spinor representation of the Lorentz algebra.

To carry out the same sort of construction in characteristic p, we first need to decide on a suitable notion of an orthogonal group. Fixing a symmetric bilinear form, η_{ab}, we can again speak of linear transformations which leave this bilinear form invariant. We refer to the corresponding Lie algebra as $\mathfrak{spin}(\eta)$. We can then construct finite-dimensional irreducible representations of the corresponding Lie algebra, which we interpret as specifying the "spin" of the corresponding physical field. This also extends to corresponding groups, which we label as $\mathrm{Spin}(\eta)$. We define a spinor sheaf as one in which for each stalk $\mathcal{S}_x$ there is a natural group action by $\mathrm{Spin}(\eta)$. It is in this sense that we are able to define spinors. From this perspective, the evaluation of each fermionic field at a point $x \in X$ should be viewed as being valued in $\mathcal{S}_x$. Again, this is quite analogous to what happens in characteristic zero.

13.1 Supersymmetry

From the way we have set up our action principle, we can even entertain a notion of supersymmetry which interchanges bosonic and fermionic degrees of freedom. Note that in lattice supersymmetry [143], there are some difficulties because finite difference operations do not respect a Leibniz rule, and this is crucial in satisfying the standard supersymmetry algebra [144]. Here, we are working in terms of general rational polynomials, and so the usual "rules of the game" for supersymmetry should carry through, at least in constructing supersymmetric actions.

To illustrate, we construct a characteristic p supersymmetric quantum mechanics. It is superficially rather close in form to the one in characteristic zero, but there are some important subtleties having to do with factors of "i".

As a warmup, we briefly review the case of $\mathcal{N} = 2$ supersymmetric quantum mechanics in characteristic zero. In that setting, the Lagrangian is:

$$L = \frac{1}{2}(\partial_t \phi)^2 + i\overline{\Psi}\partial_t \Psi + \frac{1}{2}f^2 + W' f + W'' \overline{\Psi}\Psi. \tag{13.22}$$

Here, ϕ is a real bosonic field, $\Psi = \psi_1 + i\psi_2$ and $\overline{\Psi} = \psi_1 - i\psi_2$ are complex Grassman fields, f is a real auxiliary field, and $W(\phi)$ is a superpotential. Additionally, we have adopted the standard physics convention which is to reverse the order of Grassmann variables under complex conjugation. From this, we see that the Lagrangian is invariant under complex conjugation. The action is invariant under two supersymmetry transformations (see e.g., the reviews in references [145, 146]):

$$\delta_1 \phi = i\Psi, \quad \delta_1 \Psi = 0, \quad \delta_1 \overline{\Psi} = -(\partial_t \phi + if), \quad \delta_1 f = -\partial_t \Psi \tag{13.23}$$

$$\delta_2 \phi = i\overline{\Psi}, \quad \delta_2 \Psi = -(\partial_t \phi - if), \quad \delta_2 \overline{\Psi} = 0, \quad \delta_2 f = +\partial_t \overline{\Psi}. \tag{13.24}$$

Let us verify that the action is invariant under these two transformations. Under δ_1, we have:

$$\delta_1 L = (\partial_t\phi)\,(i\partial_t\Psi) + i(-\partial_t\phi - if)\partial_t\Psi + (-\partial_t\Psi)f \tag{13.25}$$
$$+ W''(i\Psi)f + W'(-\partial_t\Psi) \tag{13.26}$$
$$+ W''(-\partial_t\phi - if)\Psi \tag{13.27}$$
$$= \partial_t\left(-W'\Psi\right), \tag{13.28}$$

which is a total derivative. Assuming suitable boundary conditions for our integral over the time coordinate, we verify that supersymmetry is a symmetry of the system. Consider next the variation under δ_2. This yields:

$$\delta_2 L = (\partial_t\phi)(i\partial_t\overline{\Psi}) - i\partial_t(-\partial_t\phi + if)\overline{\Psi} + f(\partial_t\overline{\Psi}) \tag{13.29}$$
$$+ W''(i\overline{\Psi})f + W'\partial_t\overline{\Psi} \tag{13.30}$$
$$- W''(-\partial_t\phi + if)\overline{\Psi} \tag{13.31}$$
$$= \partial_t((\partial_t\phi)(i\partial_t\overline{\Psi}) + f(\partial_t\overline{\Psi}) + W'\overline{\Psi}). \tag{13.32}$$

Note the appearance of the minus signs. This is because our convention is to only vary the "leftmost" fermionic field. We stress that nothing depends on this choice. As is standard, we can integrate out the auxiliary field f, and arrive at a physical potential for the field ϕ given by:

$$V(\phi) = \frac{1}{2}W'W'. \tag{13.33}$$

We now turn to the characteristic p version. As we already mentioned, we assume that Frobenius conjugation reverses the order of multiplication for Grassmann fields. This means that up to "some factors of i," the structure of our action should look rather similar.

With this in mind, we now consider a single $\mathbb{F}_p$ valued bosonic field $\phi(t)$ and a pair of $\mathbb{F}_p$ valued Grassmann variables $\chi(t)$ and $\psi(t)$. We could in principle introduce a "complex field" $\Psi = \chi + \widehat{i}\psi$ as well, but to track the $\mathbb{F}_p$ structure explicitly, we have chosen the current presentation. We also introduce an $\mathbb{F}_p$ valued auxiliary field $f(t)$ and a superpotential $W(\phi)$ which will be a polynomial in the ϕ variable with coefficients in $\mathbb{F}_p$. We denote the derivatives of W with respect to ϕ as W' and W''. Our proposed Lagrangian is:

$$L = \frac{1}{2}(\partial_t\phi)^2 + \widehat{i}\chi\partial_t\psi - \frac{\widehat{i}^2}{2}f^2 + W'f + \widehat{i}W''\chi\psi. \tag{13.34}$$

We now verify that this Lagrangian is supersymmetric. We introduce the two variations:

$$\delta_1\phi = \widehat{i}\psi, \quad \delta_1\psi = 0, \quad \delta_1\chi = -(\partial_t\phi + \widehat{i}f), \quad \delta_1 f = -\partial_t\psi \tag{13.35}$$
$$\delta_2\phi = \widehat{i}\chi, \quad \delta_2\psi = -(\partial_t\phi - \widehat{i}f), \quad \delta_2\chi = 0, \quad \delta_2 f = +\partial_t\chi. \tag{13.36}$$

Consider first varying with respect to δ_1. This yields:

$$\delta_1 L = (\partial_t \phi)\left(\widehat{i}\partial_t\psi\right) + \xi(-\partial_t\phi - \widehat{i}f)\partial_t\psi - \widehat{i}^2(-\partial_t\psi)f \tag{13.37}$$

$$+ W''(\widehat{i}\psi)f + W'(-\partial_t\psi) \tag{13.38}$$

$$+ W''(-\partial_t y - \widehat{i}f)\psi \tag{13.39}$$

$$= \partial_t\left(-W'\psi\right). \tag{13.40}$$

Observe that we have a "total derivative". As far as we are aware, there is no characteristic p analog of Stokes' theorem, but we shall interpret the presence of such terms as physically innocuous. Our reason for doing so is that in any sensible physical formulation, we would need to define an action modulo exact differential forms anyway, and differential forms do make sense in characteristic p.

Next, consider varying with respect to δ_2. This yields:

$$\delta_2 L = (\partial_t\phi)\left(\widehat{i}\partial_t\chi\right) - \widehat{i}\partial_t(-\partial_t\phi + \xi f)\chi - \widehat{i}^2(+\partial_t\chi)f \tag{13.41}$$

$$+ W''(\widehat{i}\chi)f + W'(+\partial_t\chi) \tag{13.42}$$

$$- W''(-\partial_t\phi + \widehat{i}f)\chi \tag{13.43}$$

$$= \partial_t((\partial_t\phi)\,\widehat{i}\chi - \widehat{i}^2 f\chi + W'\chi), \tag{13.44}$$

which is again a "total derivative." Integrating out the auxiliary field f, we arrive at a potential for the field ϕ given by:

$$V(\phi) = \frac{1}{2}W'W', \tag{13.45}$$

We can extend this analysis in a number of ways. For one, we can consider multiple fields ϕ^A, χ^A, ψ^A and f^A. Following our discussion of section 4, we can interpret this as $\mathbb{F}_q$ valued fields. In this case, the condition that we produce an $\mathbb{F}_p$ valued action is satisfied by choosing an $\mathbb{F}_q$ valued function $w(\phi)$ and then taking its norm to build the superpotential:

$$W = \prod_{i=0}^{n} F^i(w(\phi)) = w(\phi)^{1+p+\ldots+p^{n-1}} = w(\phi)^{(1-p^n)/(1-p)} \tag{13.46}$$

Observe that a critical point of W is necessarily either a zero or a critical point of $w(\phi)$. We can also introduce more general kinetic terms, much as we would in the characteristic zero setting. For example, we can write:

$$L = \frac{1}{2}K_{AB}\partial_t\phi^A\partial_t\phi^B + \widehat{i}K_{AB}\chi^A\partial_t\psi^B - \frac{\widehat{i}^2}{2}K_{AB}f^Af^B + \frac{\partial W}{\partial\phi^A}f^A + \widehat{i}\frac{\partial^2 W}{\partial\phi^A\partial\phi^B}\chi^A\psi^B. \tag{13.47}$$

13.2 A Cohomology Theory

This discussion also allows us to set up a physically motivated cohomology theory. Working on shell so that:

$$f^A = \widehat{i}^{-2} K^{AB} \frac{\partial W}{\partial \phi^B} \tag{13.48}$$

the supercharges are given by:

$$Q_+ = \widehat{i}\psi^A \left(\frac{\partial}{\partial \phi^A} + \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) = -\left(\partial_t \phi^A + \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) \frac{\partial}{\partial \chi^A} \tag{13.49}$$

$$Q_- = \widehat{i}\chi^A \left(\frac{\partial}{\partial \phi^A} - \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) = -\left(\partial_t \phi^A - \widehat{i}^{-1} \frac{\partial W}{\partial \phi^A} \right) \frac{\partial}{\partial \psi^A}, \tag{13.50}$$

where we have indicated by an explicit "derivative" (as dictated by the conjugate momentum) how it acts on a given field. We observe that both Q's are nilpotent:

$$Q_+^2 = Q_-^2 = 0 \tag{13.51}$$

and so can be used to define cohomology theories in characteristic p. In this setting, the Q's act on the space of superfield configurations. The natural grading is specified by the Fermion number, namely the number of Grassmann fields.

Now, in characteristic zero, there is a close interplay between Q-cohomology and other well known cohomological theories such as de Rham and Dolbeault cohomology. Here, the situation is quite a bit more subtle because in characteristic p, we do not have the analog of the Poincaré lemma which ensures that in suitably "small" patches, any differential form can locally be written as an exact differential form.

A reasonable analog in characteristic p to the characteristic zero de Rham cohomology goes under the name of crystalline cohomology (see e.g., [147–149]).[44] The main idea is to find a suitable way to "thicken" a characteristic p variety so as to get an analog of the Poincaré lemma. This proceeds by generating a lift of a given scheme to a characteristic zero variety. Given this, it is tempting to posit that the Q-cohomology we have just specified will work in a similar fashion.

Indeed, we note that our actual starting point for constructing physical fields began by dealing with integer valued fields, so we are free to return to this setting. Given a field ϕ taking values in $\mathbb{Z}$, we can consider its presentation in terms of a p-adic integer in $\mathbb{Z}_p$ via the formal expansion:

$$\phi = \sum_{m \geq 0} \phi_m p^m \tag{13.52}$$

in terms of the Teichmüller representatives ϕ_i (see Appendix P). To compute actual coho-

[44] In the context of the Weil conjectures, it is actually more common to consider étale and ℓ-adic cohomology theories, but on physical grounds we expect crystalline cohomology to also provide an appropriate framework as well.

mologies, we can get a "first approximation" by working modulo p. Then, we can refine this approximation by working modulo p^2, and so on. The more formal way to state this is that we view our integer valued field as specifying a Witt vector, and then addition and multiplication of physical fields is treated as the corresponding operation on $\mathbb{W}$, the space of Witt vectors (see Appendix P). There is a natural reduction mod p^n so we can also speak of $\mathbb{W}_n = \mathbb{W}/p^n\mathbb{W}$. Giving a full account of crystalline cohomology would take us too far afield. The main point for us is that in many cases of interest, we can consider a related characteristic zero scheme Z over $\mathbb{W}$. In this setting, we can indeed work in terms of de Rham cohomology, and thus obtain the relation between the crystalline cohomology of a X over a field K and its characteristic zero "cousin" Z:

$$H^i_{\text{cris}}(X/\mathbb{W}) = H^i_{\text{DR}}(Z/\mathbb{W}). \tag{13.53}$$

These cohomologies are in turn constructed via the inverse limits:

$$H^i_{\text{cris}}(X/\mathbb{W}) = \varprojlim H^i_{\text{cris}}(X/\mathbb{W}_n) \tag{13.54}$$

$$H^i_{\text{DR}}(Z/\mathbb{W}) = \varprojlim H^i_{\text{DR}}(X/\mathbb{W}_n). \tag{13.55}$$

In fact, it has also been appreciated that there are some limitations to using crystalline cohomology. One issue is that the theory makes the most sense when X is smooth and proper over a ground field K. To handle the more general situation, one often deals with a generalization known as rigid cohomology which can be applied in a more general setting [150] (see the lecture slides of reference [151] as well as the book [152]).[45] The important point for us is that this defines a universal p-adic Weil cohomology theory, and admits comparison theorems to de Rham cohomology (just like the crystalline case). Since our supersymmetric quantum mechanics formulation does not really require a smooth variety, it is tempting to conjecture that the Q-cohomology we have been dealing with specifies a crystalline cohomology in the smooth case:

$$H^i_Q(X) \simeq H^i_{\text{cris}}(X/\mathbb{W}), \tag{13.56}$$

while in the more general setting, we expect:

$$H^i_Q(X) \simeq H^i_{\text{rig}}(X). \tag{13.57}$$

Part of establishing such a correspondence will of course entail being more precise about the ring of coefficients for these different situations.

Now, in the physical theory, we often view the Q-cohomology as elements in a finite-dimensional Hilbert space. From the above considerations, it would seem natural to restrict the coefficients of this Hilbert space to a field of characteristic zero such as the one used in

[45] For a related discussion in mixed characteristic, see reference [153].

defining crystalline cohomology. The appearance of a Hilbert space also allows us to define an index for Q, as given by (see e.g. [26]):

$$\mathrm{Ind}Q = \mathrm{Tr}(-1)^{\mathbf{F}} = \mathrm{ker}Q - \mathrm{coker}Q \tag{13.58}$$

where $\mathbf{F}$ is the fermion number operator. At this point, an important comment is that even though we are dealing with a discretized spacetime and target space, we are considering all possible morphisms between these spaces, as well as their lift to formal characteristic zero spaces. For this reason we should expect on general grounds that the index of Q is in general non-trivial. This again distinguishes the present approach from lattice formulations.

By design, none of this is very different from supersymmetry in characteristic zero. Now, there is a rich mathematical story for supersymmetric quantum mechanics [154, 155]. It would be interesting to see whether this carries over to the present setting. Here we set our ambitions lower and apply this in the most simple-minded way, observing that for Wess-Zumino models, the index of Q just counts the critical points of the superpotential W, namely the locus where $dW = 0$. Geometrically, we started with a target space Y, and now can specify this subvariety as $\{dW = 0\} \subset Y$. As a a simple application, if we write $w = \widetilde{\phi}h(\phi)$ with $\{h = 0\} = Z$ a smooth subvariety of Y, then $\widetilde{\phi}$ serves as a Lagrange multiplier so that the critical points of w correspond to all the points of Z. Again, we emphasize that these notions continue to make sense in characteristic p and make no reference to any metric (which is important since we do not have a metric!). Our cohomology theories provide a convenient way to compute the resulting point set as a set of "vacua" associated with vanishing potential energy.

It is tempting to extend the analogy with the characteristic zero case even further by developing the analog of the A-model and B-model twist.[46] Here, however, we seem to have a somewhat richer set of possibilities because the way in which we complexify / adjoin roots in pass from $\mathbb{F}_p$ to some extension contains many choices. It would be interesting to study the correlators in this setting.

13.3 Zeta Functions

We now turn to a few brief comments on the connection between Zeta functions in characteristic p and our supersymmetric quantum mechanics. We introduced an index which counts (with signs) "vacua," or more precisely the critical points of a superpotential $W(\phi)$. The zero locus defines a variety V in characteristic p, and we can consider varying the ground field $\mathbb{F}_q$, which as we have seen corresponds to adding more particles into the system. From our earlier remarks on interpreting physics on $\mathbb{F}_q/\mathbb{F}_p$ for $q = p^n$ as defined by a system of n particles, we also see that there is a natural action of the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \simeq \mathbb{Z}/n\mathbb{Z}$ on this system of particles. We note that this is generated by the Frobenius map $x \mapsto x^p$.

[46] We thank E. Torres for comments on this point.

To account for this redundancy, it is appropriate to actually only count contributions to the index up to this group action. There is of course the subtlety that this group action may not act transitively on the space of solutions, but this is simply the price we pay in setting up the appropriate particle statistics. Introducing a fugacity z to track the number of particles, we introduce the more general formula:

$$\sum_{n\geq 1}\mathrm{Tr}_n\left((-1)^{\mathbf{F}}z^n\right)=\sum_{n\geq 1}\frac{\#V(\mathbb{F}_{p^n})}{|\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)|}z^n=\sum_{n\geq 1}\#V(\mathbb{F}_{p^n})\frac{z^n}{n}=\log Z_{V,p}(z), \tag{13.59}$$

which we recognize as the log of the celebrated Hasse-Weil Zeta function in characteristic p. An additional remark is that we can of course change the ground field from $\mathbb{F}_p$ to $\mathbb{F}_q$, and this also has a clear interpretation in our setting.

An additional remark here is that the Zeta function of a variety can be expressed in terms of rational function, each of which is closely tied to the characteristic polynomial for the Frobenius action on the corresponding cohomology groups. Letting $F^{(i)}: H^i \to H^i$ denote this Frobenius action, it is a linear map, and so we can construct a characteristic polynomial:[47]

$$P_i(z)=\det(\mathbf{id}-zF^{(i)}). \tag{13.60}$$

It turns out that the Zeta function can be expressed as (see [157–160]) as a superdeterminant:

$$Z_{V,q}(z)=\frac{P_1(z)...P_{2D-1}(z)}{P_0(z)...P_{2D}(z)}, \tag{13.61}$$

where D denotes the dimension of the variety V.

In Appendix K we collect a few examples of Zeta functions. In some cases, we can evaluate these expressions "by hand," but the more general case requires quite a bit more machinery. As some simple examples, we can see that in the special case where V is the affine line, we get, via our superpotential computation:

$$\#\mathbb{A}^1(\mathbb{F}_{p^n})=p^n, \tag{13.62}$$

while in the case of the projective line, we get:

$$\#\mathbb{P}^1(\mathbb{F}_{p^n})=1+p^n. \tag{13.63}$$

We observe that the Zeta function in these two cases are related to the partition functions of free particles. For example, we have:

$$Z_{\mathbb{A}^1,p}(z)=\frac{1}{1-pz} \tag{13.64}$$

[47]Our convention is as in [156]. Note also that that the natural range of i is $0,...,2D$, where D is the dimension of the variety V.

$$Z_{\mathbb{P}^1,p}(z) = \frac{1}{(1-z)(1-pz)}. \tag{13.65}$$

This basically parallels how one would expect to apply the standard Weil cohomology theories to compute the Zeta function. For example, in both the case of étale, ℓ-adic, (see [25, 161]) and rigid (see [162]) cohomology theories, one first calculates the cohomology groups $H^i(V)$ (we ignore subtleties with the coefficient ring) and then specifies the induced action of the Frobenius map $\psi : V \to V$, associated with the pullback $\psi^* : H^i(V) \to H^i(V)$. One can then count the fixed points of the Frobenius map via the associated signed index formula, namely via a formula such as:

$$\#\mathrm{Fix}(\psi) = \sum_i (-1)^i \mathrm{Tr}(\psi^*, H^i(V)). \tag{13.66}$$

Indeed, we are performing the same set of operations in our physical setting, up to one subtlety. Observe that our Q-cohomology can be viewed as specified with respect to a coefficient ring in the p-adic integers. That being said, since we are talking about computing a supersymmetric index with physical states in a standard Hilbert space, we seem to instead be referencing coefficients in $\mathbb{C}$. As we discuss in section 16, the path integral is really furnishing us with characters valued in a $\mathbb{C}$ as obtained from a "henselization" (see Appendix P) of the integers embedded in the p-adics. Because of this, the counting problems really do appear to be the same. All this is to say the usual physical strategy for computing the supersymmetric index appears to line up with its usage in the mathematical setting.

We remark that this Zeta function enters in the study of the Riemann hypothesis in characteristic p. These are connected with the development of a suitable "Weil cohomology" theory in characteristic p which has coefficients valued in characteristic zero. For a review of the Weil conjectures, see e.g., [163]. We also note that this seems to fit with one of the "Atiyah fantasies" outlined in reference [164]. Here, our choice of cohomology theory is instead specified by a choice of nilpotent supercharge.

Our proposed relation between the supersymmetric index and the Zeta function also allows us to make sense of the Zeta function, even when the variety V is singular. This seems to line up with expectations from rigid cohomology.

That being said, there are some clear pitfalls compared with the case of characteristic zero. For example, a common strategy in the characteristic zero setting is to consider perturbations in the physical theory so as to localize the path integral sum around specific field configurations. Doing so in this setting can spoil the counting problem, since for example, the Zeta function of an elliptic curve depends quite sensitively on its arithmetic properties. Of course, the failure of the index to remain invariant under such perturbations is by itself a quite intriguing feature, and points to additional structure being present in the corresponding Hilbert space.

It would be interesting to develop this further.

14 FI Parameters Revisited

Having sketched how to make sense of various field theories in characteristic p, we now turn to a potential physical application, in the context of large field ranges of a quantum field theory. The standard lore is that in a theory of quantum gravity, increasing the field range of a scalar leads to a breakdown in the low energy effective field theory. For super-Planckian field ranges, one does not expect semi-classical reasoning to carry over. In this section we revisit this class of questions from the perspective of reduction modulo p a prime number.

To keep things concrete, we focus on a 4D supersymmetric $U(1)$ gauge theory with a Fayet-Iliopoulos parameter [165]. We have already sketched how to generate a supersymmetric quantum mechanics theory, as well as gauge theories in characteristic p, so we can already anticipate that the same algebraic manipulations which are used in characteristic zero will have characteristic p analogs. It was argued in [166] that 4D $\mathcal{N} = 1$ supergravity theories without a global R-symmetry are incompatible with the existence of an FI parameter. Indeed, the typical situation in a string compactification is that such "parameters" actually arise as vevs of background fields (see e.g., [167]). Building on [14], references [15–17] argued that there is a potential loophole in such arguments if the FI parameter comes quantized in units of $2M_{\rm pl}^2$:

$$\xi = 2mM_{\rm pl}^2 \quad \text{for } m \in \mathbb{Z}. \tag{14.1}$$

where here $M_{\rm pl}$ refers to the reduced Planck mass, i.e., we have:

$$M_{\rm pl}^2 = \frac{1}{8\pi G_N}. \tag{14.2}$$

In all known string constructions, the resulting FI parameters appear to actually be "field dependent" that is, it is really just the background vev for another dynamical field. One could in principle imagine that such a large value of the FI parameter instead emerges from a suitably quantized flux. In Appendix N we present some evidence that this is indeed possible.

Here, we ask whether we can use our present perspective on field theory in characteristic p to study this and related questions where the field range becomes extremely large. We begin by writing down the bosonic sector of a 4D $\mathcal{N} = 1$ theory with gauge group $U(1)$ and chiral superfields $\varphi_1, ..., \varphi_n$ with charges $q_1, ..., q_n$. Anomaly cancellation imposes the conditions $q_1 + ... + q_n = 0$ and $q_1^3 + ... + q_n^3 = 0$ (namely cancellation of $U(1)\text{grav}^2$ and $U(1)^3$ anomalies), but one can in principle relax these conditions by viewing the gauge theory as a subsector of a bigger model. In characteristic zero, the bosonic sector of the Lagrangian contains the terms:

$$S \supset \int d^4x \, \left(-\frac{1}{4g^2} F_{ab}F^{ab} + \sum_i \left|\partial\varphi_i + q_i A\varphi_i\right|^2 - \frac{g^2}{2} \left(\sum_i q_i \left|\varphi_i\right|^2 - \xi \right)^2 \right), \tag{14.3}$$

where g refers to the gauge coupling of the $U(1)$ gauge theory. We now consider performing a similar rescaling as that indicated in section 3. We assume that each field can move a minimal step Λ_{min}. We also make the replacement $d^4x \mapsto \Lambda_{\text{max}}^{-4}$ and $\partial \mapsto \Lambda_{\text{max}}\partial$. Focussing on just the terms which involve the scalar field (we can include the gauge field kinetic term in much the same way), our proposal for a discretized action is:

$$S_{\text{disc}} = \sum_{x\in X}\left(\frac{\Lambda_{\text{max}}^2}{\Lambda_{\text{min}}^2}\sum_i |\mathcal{D}_a\varphi_i|^2 - \frac{g^2}{2}\frac{\Lambda_{\text{min}}^4}{\Lambda_{\text{max}}^4}\left(\sum_i q_i\,|\varphi_i|^2 - 2m\frac{M_{\text{pl}}^2}{\Lambda_{\text{min}}^2}\right)^2\right). \tag{14.4}$$

We make the assumption that the ratios of energies come in quantized steps so that we can set:

$$\frac{\Lambda_{\text{max}}^2}{\Lambda_{\text{min}}^2} = \frac{2\pi}{N},\quad \frac{M_{\text{pl}}^2}{\Lambda_{\text{min}}^2} = M,\quad \frac{g^2}{2}\frac{\Lambda_{\text{min}}^4}{\Lambda_{\text{max}}^4} = \frac{2\pi}{N}B. \tag{14.5}$$

for some integers $B, K, N \in \mathbb{Z}$. The factors of π appearing here are actually rather natural. For example, we can also present these conditions as:

$$\frac{\Lambda_{\text{max}}^2}{\Lambda_{\text{min}}^2} = \frac{2\pi}{N},\quad \frac{M_{\text{P}}^2}{\Lambda_{\text{min}}^2} = 8\pi K,\quad \alpha_{U(1)}\frac{\Lambda_{\text{min}}^4}{\Lambda_{\text{max}}^4} = \frac{B}{N}, \tag{14.6}$$

where M_{P} is the non-reduced Planck mass, i.e., $M_{\text{P}}^2 = G_N^{-1}$ and $\alpha_{U(1)} = g^2/4\pi$. In any event, this motivates us to consider the discretized action:

$$S_{\text{disc}} = \frac{2\pi}{p}\sum_{x\in X}\left(\sum_i |\mathcal{D}_a\varphi_i|^2 - B\left(\sum_i q_i\,|\varphi_i|^2 - 2r\right)^2\right), \tag{14.7}$$

namely, we work modulo $N = p$ a prime number, and we have introduced an integer parameter $r = mK$. Here, we have also assumed that our fields are valued in $\mathbb{F}_p(\widehat{i})$, and the $|\cdot|$ notation refers to expanding out as a square, i.e.:

$$|\varphi|^2 = a^2 - \widehat{i}^2 b^2 \quad \text{for} \quad \varphi = a + \widehat{i}b \quad \text{with } a, b \in \mathbb{F}_p. \tag{14.8}$$

In terms of our previous conventions where we view all physical fields as rational morphisms between schemes, we also absorb the factor of $\hbar = p/2\pi$ into our definition of the path integral phase. In terms of this, we reach our characteristic p action:

$$S = \sum_{x\in X}\text{ev}_{u=x}\left(\sum_i |\mathcal{D}_a\varphi_i|^2 - B\left(\sum_i q_i\,|\varphi_i|^2 - 2r\right)^2\right), \tag{14.9}$$

In this case, we can apply all the machinery previously developed. One immediate observation is that in working mod p, it could happen that the discretized analog of the FI parameter now vanishes. So, an expansion around $\xi = 0$ and a super-Planckian FI parameter can in such cases appear quite similar in characteristic p.

Let us now turn to the vacua of the system. In characteristic zero we label these as zeros of the effective potential, modulo $U(1)$ gauge transformations. This defines a toric variety Y and the procedure just outlined specifies a symplectic quotient:

$$Y = (\mathbb{C}^*)^n //U(1). \tag{14.10}$$

One can generalize this in various ways by including additional fields, as well as multiple $U(1)$ factors. Note that because we demanded that the q_i's sum to zero, we have a toric Calabi-Yau space, but there are some additional constrains which are being imposed from working in four dimensions. If we had considered the analogous problem in a 2D system, we could relax these conditions further. For additional discussion of the 2D field theory analysis, see e.g., [168].

With this in mind, we now consider the analogous class of questions associated with 2D gauged linear sigma models with $\mathcal{N} = (2, 2)$ supersymmetry. In this setting, the connection to toric geometry becomes quite apparent. Two canonical examples of non-compact Calabi-Yau spaces which are captured by such a symplectic quotient include $\mathcal{O}(-n) \to \mathbb{CP}^{n-1}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. With suitable charge assignments, a positive value of the FI parameter specifies the volume of the compact $\mathbb{CP}^{n-1}$ and $\mathbb{CP}^1$ factor. In the case of the conifold, switching to negative values of the FI parameter signals a flop transition.

Consider next the related analysis in characteristic p. See for example [169] for some discussion of toric geometry in characteristic p. In this case, we can still define an appropriate quotient by a group action, but now it is of the form:

$$Y = \left(\mathbb{F}_p(\widehat{i})\right)^n //U(1, \mathbb{F}_p(\widehat{i})). \tag{14.11}$$

As an illustrative example, in the case of $\mathcal{O}(-n) \to \mathbb{P}^{n-1}$, the D-term equation can be written as:

$$|\varphi_1|^2 + |\varphi_2|^2 + ... + |\varphi_n|^2 - n\,|z|^2 = 2r. \tag{14.12}$$

with fields φ_i of charge $+1$ and z of charge $-n$, with solutions identified modulo the group action by $U(1, \mathbb{F}_p(\widehat{i}))$. Suppose we fix the value of z. In the characteristic zero setting, this would define a $\mathbb{P}^{n-1}$, and we expect something similar to hold in characteristic $p > 0$ as well. To see why, let us begin by introducing a copy of $\mathbb{P}^{n-1}(\mathbb{F}_p(\widehat{i}))$ with homogeneous coordinates $[u_1, ..., u_n] \sim [\lambda u_1, ..., \lambda u_n]$ for $\lambda \in \mathbb{F}_p(\widehat{i})$. A convenient way to parameterize the $\lambda \in \mathbb{F}_p(\widehat{i})$ is in terms of a "radial part" and a "phase":

$$\lambda = c \cdot \mu, \tag{14.13}$$

for $c \in \mathbb{F}_p$ and $\mu \in U(1, \mathbb{F}_p(\widehat{i}))$. Indeed, working over the ground field $\mathbb{F}_p$, we can build a $U(1, \mathbb{F}_p(\widehat{i}))$ bundle over $\mathbb{P}^{n-1}$, and the total space is of dimension $2n-1$ over the ground field

$\mathbb{F}_p$. This is the characteristic p version of constructing an S^{2n-1} as a circle bundle over $\mathbb{CP}^{n-1}$. Just as there, the affine quadric specified by the D-term constraint of line (14.12) provides a convenient way to build an $S^{2n-1}(\mathbb{F}_p)$. So, working modulo $U(1,\mathbb{F}_p(\widehat{i}))$ identifications, we expect that the symplectic quotient specified by equation 14.11 does indeed define a $\mathbb{P}^{n-1}(\mathbb{F}_p(\widehat{i}))$.

An interesting feature of this analysis is that provided $2r+n\left|z\right|^2$ is non-zero, the number of points in this $\mathbb{P}^{n-1}$ is always the same, even if we vary the FI parameter. To see this, we note that our $\mathbb{P}^{n-1}$ can instead be written as a coset space:

$$\mathbb{P}^{n-1}=\frac{SU(n,\mathbb{F}_p(\widehat{i}))}{SU(n-1,\mathbb{F}_p(\widehat{i}))\times U(1,\mathbb{F}_p(\widehat{i}))}, \tag{14.14}$$

and so given a single point in this space, we can use the transitive $SU(n,\mathbb{F}_p(\widehat{i}))$ group action to reach any other point.

We can also count the number of such points. To do this, suppose that in the presentation in terms of the homogeneous coordinates $[\Phi_1,\Phi_2,...,\Phi_n]$ the coordinate Φ_1 is non-zero. Then, we get an affine patch with coordinates Φ_i/Φ_1 for $i=2,...,n$. There are $n-1$ such coordinates, so we get a total of p^{n-1} distinct points. Next, suppose that $\Phi_1=0$ but that Φ_2 is non-zero. In this patch, we have coordinates Φ_i/Φ_2 for $i=3,...,n$, and we get a total of p^{n-2} distinct points. Continuing in this way, we can get all the way down to all $\Phi_i=0$ for $i=1,...,n-1$, and we are left with the single point $[0,...,1]$, which counts as just one point. The total number of points is then:

$$\left|\mathbb{P}^{n-1}\right|=p^{n-1}+p^{n-2}+...+p+1=\frac{1-p^n}{1-p}. \tag{14.15}$$

We note that this is essentially a decomposition of $\mathbb{P}^{n-1}$ into smaller constituents:

$$\mathbb{P}^{n-1}=\mathbb{A}^{n-1}\oplus\mathbb{A}^{n-2}\oplus...\oplus\mathbb{A}^1\oplus\mathbb{A}^0. \tag{14.16}$$

More generally, consider varying the value of $r'=2r+n|z|^2$. The D-term constraint can then be presented as:

$$\left|\varphi_1\right|^2+\left|\varphi_2\right|^2+...+\left|\varphi_n\right|^2=r', \tag{14.17}$$

modulo $U(1,\mathbb{F}_p(\widehat{i}))$ transformations. If there were no symplectic quotient, we would just get a copy of $\mathbb{A}^n$, which has p^n points. Varying r' over $\mathbb{F}_p$, we see that there are $(p-1)$ non-zero values, and one where it vanishes. So, the total number of points in $\mathbb{A}^n$ can be written as:

$$\left|\mathbb{A}^n\right|=(p-1)\left|\mathbb{P}^{n-1}\right|+\left|\mathbb{P}^{n-1}_{r'=0}\right|, \tag{14.18}$$

so we also learn that:

$$\left|\mathbb{P}^{n-1}_{r'=0}\right|=p^n-(p-1)\frac{1-p^n}{1-p}=1. \tag{14.19}$$

Consequently, if we now vary z, we see that we can visualize the total space as a collection of $\mathbb{P}^{n-1}$ spaces. These split into two types, ones where $n\,|z|^2 + 2r \neq 0$, and those with $n\,|z|^2 + 2r = 0$. We note that both can occur "frequently" when in characteristic p. Another comment is that even as we vary the FI parameter, we do not recover a "macroscopic geometry." Indeed, each of our shells has a finite number of points.

As another example, consider the case of a conifold, namely a quadric. From the perspective of our $U(1)$ gauge theory, we introduce two charge $+1$ fields u_1, u_2, and two charge -1 fields v_1 and v_2. The point set is then captured by the condition:

$$|u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 = 2r, \tag{14.20}$$

modulo identifications by the $U(1, \mathbb{F}_p(\widehat{i}))$ group action. Here, we observe another curiosity: In characteristic p, the notion of r "positive and negative" does not really make sense. Of course, the action $r \to -r$ does still switch the roles of the u and v coordinates, corresponding to a flop transition, but we can no longer identify this with just continuing a Kähler class to negative values.

15 Geometric Engineering in Characteristic $p > 0$

As already mentioned in the Introduction, one of the motivations for the present work is that some of the main tools used in constructing string vacua make use of techniques from algebraic geometry, with little explicit use made of the actual spacetime metric in the extra dimensions. This is usually viewed as a problem, because it means that many non-holomorphic quantities of interest such as the masses of particles can at best be obtained in some approximation scheme. On the other hand, the very fact that these constructions are often formulated in the algebraic setting is a *welcome* feature in studying the passage to characteristic p. Based on our physical considerations presented in section 3, we view the resulting arithmetic geometries as the highly quantum regime of a string compactification. This by itself is rather intriguing and seems motivation enough. Here we consider a variant of geometric engineering [27–30] but in characteristic p.

Our plan in this section will be to make use of some of the more rigorously established aspects of geometric engineering in characteristic zero, now transported to the characteristic $p > 0$ setting. Our string compactification geometries will be Calabi-Yau varieties. This means the canonical sheaf is trivial. We will be interested in geometric engineering, the framework used to connect certain singular string compactification geometries to partially twisted field theories. For a recent overview of geometric engineering in characteristic zero, we refer the interested reader to reference [170]. For some recent discussion of Calabi-Yau spaces over finite fields, see e.g., [79, 80, 76].

15.1 Higgs Bundles and Local Singularities

The first non-trivial example we wish to consider involves a correspondence between the Hitchin system on a genus g complex curve Σ with an ADE gauge group G, and a local singular Calabi-Yau threefold Y comprised of a curve Σ of ADE singularities.

For example, in the case of an A_{M-1} singularity, the singularity can be presented as the hypersurface equation:

$$xy = z^M, \tag{15.1}$$

where $(x = y = z = 0)$ denotes the location of the curve. We will denote by Y_t the smoothings of the threefold. The physics of this system has been investigated in a number of papers, for a partial list of examples see e.g. [29, 171–176]. Let us briefly review the match between the two moduli spaces, working at smooth points.

The correspondence involves matching the Hitchin moduli space to the moduli space defined by the Weil intermediate Jacobian of the Calabi-Yau. In this correspondence, the base of the Hitchin moduli space defined by Casimir invariants of a Higgs field maps to smoothing deformations of the Calabi-Yau. The data of holonomies in the Hitchin system maps to periods of a three-form potential defined on the Calabi-Yau threefold. On the Hitchin system side of the story, we specify a pair $(\mathcal{E}, \Phi)$ consisting of a principal G bundle

$\mathcal{E}$ and a Higgs field Φ, which defines a map:

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}_\Sigma, \tag{15.2}$$

with a suitable notion of stability for the Higgs bundle. The first important match between these two structures is the mapping between coefficients in the spectral equation of the Higgs field (viewed as a hypersurface in the canonical bundle over Σ) and smoothing deformations of the local Calabi-Yau threefold. In more detail, recall that the spectral equation for the Higgs field in the fundamental representation is:

$$\det(u\mathbb{I}_{M\times M} - \Phi) = \sum_{i=0}^{M} c_i u^{M-i} = 0, \tag{15.3}$$

with u a section of the canonical bundle, and c_i a Casimir invariant built from the Higgs field, which we view as a section of $(\mathcal{K}_\Sigma)^{\otimes M}$. These map to unfoldings of the singularity:

$$xy = \sum_{i=0}^{M} c_i u^{M-i}. \tag{15.4}$$

The zero set of the spectral equation specifies a spectral cover of the original curve:

$$\widetilde{\Sigma} \xrightarrow{\pi} \Sigma. \tag{15.5}$$

Additionally, we can equip $\widetilde{\Sigma}$ with a line bundle $\widetilde{\mathcal{L}}$, and via the spectral cover construction [177], the push-forward map under π_* generates a vector bundle. This line bundle can also be viewed as being specified by a point in the Jacobian of $\widetilde{\Sigma}$, and this in turn has a direct analog in the smoothed Calabi-Yau threefold geometry Y_t as a point in the Weil intermediate Jacobian $\mathcal{J}(Y_t)$.

An important feature of establishing this correspondence rigorously is that it can actually be formulated *algebraically*, with no direct reference to metric data.[48] Given everything we have seen so far, it would seem natural to expect a correspondence over characteristic p to also hold. Again working with respect to the A-type case, we expect that on the Hitchin system side of the correspondence will involve an $SL(N, \mathbb{F}_q)$ vector bundle $\mathcal{E}$, and a Higgs field

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}_\Sigma. \tag{15.6}$$

Encouragingly, we note that some work has been done on developing Hitchin systems in characteristic p, and has even figured in the proof of the Fundamental Lemma of the Langlands program (see e.g., [178–180]), and this fits in the broader discussion of formulating Higgs bundles in characteristic p. A natural question to address in this direction would

[48]We thank R. Donagi and T. Pantev for discussions on this point.

be to develop suitable comparison theorems from changing the ground field.[49] Presumably, the closest analog to the characteristic zero correspondence holds for the case of $\overline{\mathbb{F}}_p$, but we expect that the more general situation over a finite field is also well-defined.

There are various generalizations of this basic correspondence. In characteristic zero, it is also expected that we can instead consider a Kähler manifold S of dimension d equipped with a Higgs bundle with structure group G an ADE group specified by the pair $(\mathcal{E}, \Phi)$. In this case, the expectation is that there is again a correspondence, but this time with a local Calabi-Yau $(d+2)$-fold Y specified by a Kähler manifold of ADE singularities (see e.g., [171, 184–186]). A non-trivial feature of the $d > 1$ case is the appearance of non-zero "bulk fluxes" on the Calabi-Yau side, which in turn is expected to be specified (at a suitable smoothing of the singular Calabi-Yau) by the Deligne-Beilinson cohomology [187, 172]. While even this has not been established in full generality, one expects that an algebraic correspondence will also be available in this case as well.

Indeed, we observe that part of this correspondence is straightforward to establish, both in characteristic zero and in characteristic p. For ease of exposition, we assume that the canonical bundle of S is very ample. In this case, we can construct the spectral equation in the total space of the canonical bundle for S, and match the corresponding Casimir invariants with smoothing deformations of the Calabi-Yau Y. In all these cases, we expect that this extends to meromorphic Higgs fields with singularities specified along various subspaces. As a particular case of interest, observe that we can now specify a characteristic p version of the Vafa-Witten system [188] on a Kähler surface. This in turn suggests a potential way to connect with the GL twist of reference [189], though in the geometric engineering setting, this is usually not phrased as a purely holomorphic problem.[50] For example, recall that in characteristic zero, we can engineer $\mathcal{N} = 4$ Super Yang-Mills theory by working with type IIB strings on a $\mathbb{E} \times \mathbb{C}^2/\Gamma_{ADE}$, with $\mathbb{E}$ an elliptic curve and $\mathbb{C}^2/\Gamma_{ADE}$ an ADE singularity, as defined by a singular hypersurface equation. This sort of geometry still makes sense in characteristic p, so presumably we can use this to set up a characteristic p analog of reference [189].

One difficulty we encounter in the characteristic p setting is that while there is a match between smoothing deformations of the local Calabi-Yau spaces and deformations of the spectral equation for the Higgs bundle, the extension of this to include the data of Deligne-Beilinson cohomology is less straightforward. In particular, a point emphasized in reference [172] is that in the characteristic zero setting for Calabi-Yau threefolds, the behavior of limiting mixed Hodge structures plays a crucial role in matching the data of the Weil intermediate Jacobian of the Calabi-Yau threefold to the corresponding vector bundle data of the Higgs bundle specified on a Hitchin system. But we have also seen in section 13 that

[49] As a recent example of how this ought to work, see e.g., references [181–183].

[50] Rather, the local model in question involves branes wrapped on the local Calabi-Yau fourfold T^*M_4 with M_4 a four-manifold. It would clearly be interesting to determine whether a characteristic p version of this story makes sense, particularly sense the GL twist figures prominently in the geometric Langlands program [189].

the structure of rigid cohomology enters in a structural way in our discussion of systems with supersymmetry, especially with regards to using the lifting to the ring of Witt vectors of the finite field. Doing so, we obtain a characteristic zero variety, and in that setting, we can implement a p-adic analog of Deligne-Beilinson cohomology known as syntomic cohomology [190,191]. In that setting, we have two filtrations, as associated with the Frobenius map and the Hodge filtration structures on crystalline cohomology. The combined Hodge and Frobenius structures produce a filtered $\phi_{\text{Frobenius}}$-module which plays the p-adic analog to the Hodge structure given in the characteristic zero setting. Syntomic cohomology is then derived from the Hodge and Frobenius structures, much as Deligne-Beilinson cohomology is derived from the Hodge structure.[51]

15.2 6D SCFTs and Topological Modular Forms

A fruitful approach to the study of quantum fields engineered via string theory is to use six-dimensional superconformal field theories (6D SCFTs) as a starting point for generating (via further compactification) a number of lower-dimensional quantum field theories. For a recent review of 6D SCFTs and how they are engineered in F-theory via elliptically fibered Calabi-Yau threefolds, see references [192,193]. Since we can equally well define such Calabi-Yau spaces over different ground fields, this immediately provides an operational definition of what one would mean for at least the Higgs branch moduli space, since this is controlled by deformations of the defining equations (viewing the Weierstrass model for the elliptically fibered Calabi-Yau as cutting out a possibly singular hypersurface in some ambient projective variety).[52] One recent intriguing observation is that the compactification of a 6D SCFT on a four-manifold provides a general template for realizing two-dimensional theories with minimal $\mathcal{N} = (1,0)$ supersymmetry (see e.g., [194–196]). Now, as conjectured in [197,198], topological modular form cohomology classes can be understood in terms of 2D supersymmetric quantum field theories. Thus, the compactification of 6D SCFTs on four-manifolds provides a route to generating topological modular forms [196].[53] In particular, the elliptic genus of the 2D theory [204] specifies a four-manifold invariant, and thus relates the theory of topological modular forms to four-manifolds [196].

Now, since we can also specify characteristic p analogs of four-manifolds, there is a natural operation we can consider whereby we take the original base $\mathcal{B}$ of an elliptically fibered threefold and replace it by a fibration $\mathcal{B} \to M_4$ over our four-manifold. Next, interpret this as the base of an elliptically fibered fivefold. So long as the total space for the fibration $\mathcal{B} \to M_4$ is specified by an algebraic equation, we can again introduce a Calabi-Yau fivefold,

[51] We thank D. Corwin for helpful clarifying comments on these points, some of which we have added here, essentially verbatim.

[52] The tensor branch is specified by blowups of the base, and so it is not altogether clear whether this notion will survive in the characteristic p setting.

[53] For surveys of topological modular forms, see e.g, [199, 200], and for related physical discussions see e.g., [196, 201–203]. It is of course also tempting to extend these considerations to higher chromatic type.

and this will generate 2D $\mathcal{N} = (0,2)$ theories. Taking the two-dimensional "spacetime" of this effective field theory to be an elliptic curve, we see that the Weierstrass model of our Calabi-Yau fivefold is more singular at primes $p = 2, 3$. This is perhaps correlated with the fact that $\mathbf{TMF_3} \simeq \mathbb{Z}/24\mathbb{Z}$. In any event, it would seem interesting to consider the mod p reductions of various 6D SCFT backgrounds, as well as their subsequent compactifications.

15.3 Non-Calabi-Yau Cases

In the context of string compactification, Calabi-Yau spaces amount to a special choice because they preserve some supersymmetry in the uncompactified direction. This is because Calabi-Yau spaces admit covariantly constant spinors. As is well-known, this can be generalized in various ways. First of all, one can relax the condition of $SU(n)$ metric holonomy for a complex n-fold, and instead only demand the existence of an $SU(n)$ structure group for the tangent bundle. This is especially prominent in the study of backgrounds with fluxes switched on, and the resulting "uncompactified" direction is (at least in controlled examples) an Anti-de Sitter space background. Insofar as we can start from a Calabi-Yau background and then switch on such fluxes, we expect to capture such data by the choice of a background set of quantized fluxes, i.e., as captured by elements in $H^i(X, \mathbb{Z})$. This approach has actually led to a number of recent insights in the arithmetic structure of flux vacua, see e.g., [72, 74, 83, 84, 86]. In this sense, the program we have been advocating amounts to a physical justification for the procedure of considering a string compactification "mod p".

But there are also additional ways in which a string compactification can preserve supersymmetry, to say nothing of the possibility of non-supersymmetric backgrounds and their possible extra-dimensional geometric origins (whatever this may be). Here, we would like to explain how to extend our considerations to such situations. Again, we find it helpful to proceed by way of example rather than offer a single overarching prescription. To begin, let us recall the characteristic zero case of a local seven-manifold with metric holonomy G_2, as specified by the Bryant Salamon space [205]. This is given by a round S^3 equipped with the left-handed spinor bundle $\mathbb{S}_L$. The resulting total space X is the fibration $\mathbb{S}_L \to X \to S^3$. Now, the important point for us is that this space can also be written as a real algebraic equation. Indeed, introducing complex coordinates $z_j = x_j + \sqrt{-1}y_j$ for $j = 1, 2, 3$, we have:

$$\mathrm{Re}\left(\sum_{j=1}^{4}(z_j)^2\right) = \mu, \tag{15.7}$$

for $\mu \in \mathbb{R}$. In terms of the real coordinates x_j and y_j, we have:

$$\sum_{j=1}^{4}(x_j)^2 = \mu + \sum_{j=1}^{4}(y_j)^2, \tag{15.8}$$

and for $\mu > 0$, the minimal size S^3 of radius $\sqrt{\mu}$ sits at $y_j = 0$, while for $\mu < 0$, the minimal size S^3 of radius $\sqrt{-\mu}$ sits at $x_j = 0$.

While this construction seems quite tied to special properties of the real differentiable structures, we observe that in equation (15.7), there is the closely related non-compact Calabi-Yau threefold given by the smoothing deformation of the singular quadric (i.e., the deformed conifold):

$$\sum_{j=1}^{4} (z_j)^2 = \mu. \tag{15.9}$$

Assuming $\mu > 0$, we see that this parameter sets the size of the base S^3 in the geometry $T^* S^3$, as specified by the real equations:

$$\sum_{j=1}^{4} (x_j)^2 = \mu + \sum_{j=1}^{4} (y_j)^2 \text{ and } \sum_{j=1}^{4} x_j y_j = 0. \tag{15.10}$$

Returning to equation (15.9), this defining space is Calabi-Yau, and can be formulated over $\mathbb{C}$, but any other ground field, including for example $\overline{\mathbb{F}}_p$, provided we now interpret $\mu \in \overline{\mathbb{F}}_p$, where all notions of "big and small" are again somewhat meaningless. In particular, we can still speak of the deformed Calabi-Yau, as specified by equation (15.9).

Let us now turn to the generalization to a suitable notion of a characteristic p G_2 space. Observe that in the case of the G_2 space over $\mathbb{R}$, this can be interpreted as the set of points invariant under complex conjugation under $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$. By a similar token, we can introduce the q^{th} Frobenius conjugation $z_j \mapsto z_j^q$ on each coordinate. Formally, we can then introduce the trace over the Frobenius map, and then consider the resulting point set. A difficulty we face in this procedure is that whereas $\mathbb{C}$ is a finite extension of $\mathbb{R}$, $\overline{\mathbb{F}}_p$ is a field extension with infinite degree over $\mathbb{F}_q$. So, the best we can do is introduce some representative q' such that $\mathbb{F}_{q'}$ is a finite field extension of $\mathbb{F}_q$. Then, we can write:

$$\mathrm{Tr}_{\overline{\mathbb{F}'_q}/\mathbb{F}_q} \left(\sum_{j=1}^{4} (z_j)^2 \right) = \mu, \tag{15.11}$$

where now we have restricted to $\mu \in \mathbb{F}_q$. More explicitly, we have:

$$\left(\sum_{j=1}^{4} (z_j)^2 \right) + \left(\sum_{j=1}^{4} (z_j)^2 \right)^q + ... + \left(\sum_{j=1}^{4} (z_j)^2 \right)^{q^{d-1}} = \mu, \tag{15.12}$$

where d denotes the degree of the field extension $\mathbb{F}_{q'}$ over $\mathbb{F}_q$. Insofar as we are simply adding in additional points, we can consider the formal limit obtained by proceeding to $\overline{\mathbb{F}}_p$, but in this case one would strictly speaking be dealing with an infinite number of equations and variables over $\mathbb{F}_q$.

Following this sort of procedure, we see that whenever we can specify a G_2 space in terms of a real algebraic variety, there is a corresponding generalization to characteristic p geometry. Of course, there are limitations to this approach because in many cases the precise form of these algebraic equations is not known for G_2 spaces! That being said, even the existence of such a formulation shows that we can make sense of more delicate structures such as G_2 spaces. An additional comment here is that similar considerations hold for $Spin(7)$ spaces, and in some sense this case is "easier" because in the real setting it has the same real dimension as a Calabi-Yau fourfold.

With all of this in place, we see that much of what we said about geometric engineering in the Calabi-Yau setting can be extended to this more class of geometries. Indeed, there has been recent progress in understanding the corresponding between manifolds of ADE singularities and their corresponding characterization in terms of a partially twisted field theory (see e.g., [206–213]). The main bottleneck, then, is simply that explicit presentations of the requisite real algebraic varieties is not at present known, namely it is a technical (albeit a non-trivial and important one), rather than a conceptual difficulty.

Part III

Mixed Characteristic and Beyond

16 Lifting to the p-adics

In the previous sections we showed that there is a rich geometric structure present in the special case where we take the reduced Planck constant to be $p/2\pi$ for p a prime number. Indeed, this allowed us to build up a formulation of physical fields based on geometry in characteristic p. With the aim of eventually understanding how to make sense of the general case:

$$\hbar = \frac{N}{2\pi}, \tag{16.1}$$

we now consider a somewhat more general case where where $N = p^a$. Our plan will be to see how to recognize the appearance of emergent topological structures in this limit. In particular we will show that this leads to a physical formulation over geometries in mixed characteristic, namely the ground fields will be characteristic zero, but their residue fields will be of characteristic $p > 0$. In section 21 we extend these considerations even further to cover the case of N given by $N = p_1^{a_1}...p_m^{am}$.

Now, since our starting point was a basis of fields with integer coefficients, we are free to consider a p-adic expansion for any such integer t of the form:[54]

$$t = \sum_i t_i p^i. \tag{16.2}$$

Working in terms of polynomials in $\mathbb{Z}_p[u_1, ..., u_D]$, with $\mathbb{Z}_p$ the ring of p-adic integers, we see that we can also construct an action, and perform a similar p-adic expansion:

$$S = \sum_{j \geq 0} S_j p^j, \tag{16.3}$$

which truncates at finite order (for a given field configuration). Reduction modulo N means that we simply drop the higher order terms in this expansion.

This p-adic expansion also shows that at least in the limit $a \to \infty$, there is a natural topology for our basis of fields and our action. To see how it comes about, we observe first that for each $n \in \mathbb{N}$, the space $\mathbb{Z}/p^n\mathbb{Z}$ can be equipped with the discrete topology (each point is both open and closed). Next, we can view the p-adic integers $\mathbb{Z}_p$ as obtained from the inverse limit (see Appendix I):

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}. \tag{16.4}$$

Consequently, we can equip $\mathbb{Z}_p$ with the relative product topology. This turns out to generate the same topology as we would get if we had just introduced the p-adic norm $|\cdot|_p$ from the

[54] In mixed characteristic we do not need to tread as carefully in how we refer to "local coordinates" t and u versus "evaluation points" x, and so we will freely conflate the different choices so long as the context is clear.

start. Recall that for an integer $t = p^n t'$ with t' relatively prime to p, we have:

$$|t|_p = p^{-n}. \tag{16.5}$$

So, higher order powers in p are actually small corrections! Building the field of fractions out of $\mathbb{Z}_p$, we reach the p-adic numbers $\mathbb{Q}_p$, and the p-adic norm extends in the expected way (e.g. $|p^n|_p = p^{-n}$). The norm is non-Archimedean in the sense that it satisfies a much stronger form of the triangle inequality. For $t, t' \in \mathbb{Q}_p$, we have:

$$|t + t'|_p \leq \max(|t|_p, |t'|_p) \leq |t|_p + |t'|_p. \tag{16.6}$$

Moreover, if $|t|_p \neq |t'|_p$, then we have the equality $|t + t'|_p = \max(|t|_p, |t'|_p)$. Clearly, this is a different notion of proximity from what one is accustomed to dealing with in using the real numbers, but if all we consider is the integers, there is a priori no issue with introducing such a norm.[55]

Much as in our discussion of characteristic $p > 0$ geometries, we will also be interested in cases where the ground field is taken to be a field extension L of $\mathbb{Q}_p$. The p-adic norm extends to this more general setting since, if the degree of the field extension is $n = [L : \mathbb{Q}_p]$, we can take the Norm in the sense of Galois theory (i.e., the product over all Galois conjugate values) to extend the p-adic norm for any $\alpha \in L$:

$$|\alpha|_p = |\mathrm{Norm}_{L/\mathbb{Q}_p}(\alpha)|_p^{1/n}. \tag{16.7}$$

In all cases, L admits an analogous π-adic expansion which is related to p, but the precise nature of this expansion parameter depends on the degree of ramification in the extension.[56]

Returning to the case where the ground field is $\mathbb{Q}_p$, the specification of the coefficients t_j and S_j in equations (16.2) and (16.3) is actually somewhat subtle. One's first inclination might to be just fix the coefficients according to coefficients valued in $\{0, ..., p-1\}$, with each of these having a clear interpretation in $\mathbb{F}_p$. This turns out to only work for the leading order coefficients. The main issue is that we would like to have a suitable notion of coefficient-wise addition and multiplication so that we need not worry about "carry over" from arithmetic operations. The problem is solved by working with Teichmüller representatives. At a practical level, this involves using p-adic coefficients which satisfy the relation $\omega^p = \omega$ in $\mathbb{Z}_p$. A very non-trivial feature of the ring of integers is that all solutions to this equation are actually elements of $\mathbb{Z}_p$.

[55]Observe also that for any rational number $t \in \mathbb{Q}$, there is an adelic relation by taking the product over all primes $\prod\limits_p |t|_p = \frac{1}{\mathrm{t}_{\mathbb{R}}}$, so in this sense, knowing "enough about t at all primes" provides a reconstruction of the corresponding real norm.

[56]Given L a field extension of K, we can consider the ring of integers $\mathcal{O}_L$ and $\mathcal{O}_K$, and consider the algebraic closure of a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ in $\mathcal{O}_L$. The resulting factorization into primes of $\mathcal{O}_L$, namely $\mathfrak{p}\mathcal{O}_K = \mathfrak{p}_1^{e_1} ... \mathfrak{p}_n^{e_n} \mathcal{O}_L$. We say the extension is unramified at the prime $\mathfrak{p}$ if all e_i are 0 or 1, and ramified otherwise. See Appendix S for additional discussion.

This is, in fact, part of the more general line of development associated with the ring of Witt vectors for a characteristic p field, something we review in Appendix P. The key point is that for components of two Witt vectors U and V, we do have a natural mod p^{i+1} relation of the form:

$$(U+V)_i = U_i + V_i \tag{16.8}$$

$$(UV)_i = U_i V_i. \tag{16.9}$$

For our purposes, we mainly need to apply this formalism in the case of the p-adic integers $\mathbb{Z}_p$. In that setting we can present each field ϕ as well as the action in terms of its Teichmüller representative. In this more abstract setting, we can now work with schemes defined over $\mathbb{Z}_p$ reduced modulo p^a.

With this in place, we can now set up a very similar line of development to what we initially considered in the case of physics over finite fields of characteristic p. In this case, we can choose to consider bosonic physical fields as locally specified by polynomials in $\mathbb{Z}_p[u_1, ..., u_D]$. All of the geometric flavor introduced previously still appears to make sense, provided we interpret our geometric structures as varieties over the ground field $\mathbb{Q}_p$ reduced modulo p^a, or even better, as schemes over $\mathbb{Z}_p$ reduced modulo p^a. This latter feature also illustrates that restricting to just polynomials with $\mathbb{Z}_p$ coefficients should provide a suitable notion of "convergence" of these power series. Note that there is also a natural boundary which emerges in these geometries, since $\mathbb{Z}_p$ consists of elements with p-adic norm less than or equal to one (i.e., it specifies a disk).

Instead of dealing with the ring of Witt vectors for $\mathbb{F}_p$, we can instead consider the ring of Witt vectors for $\mathbb{F}_q$, a degree n field extension of $\mathbb{F}_p$. When we do so, the same procedure just outlined produces a variety over $\mathbb{Q}_q$, the degree n unramified extension of $\mathbb{Q}_p$. In this case, we can again reference a ring of integers $\mathbb{Z}_q$, but now there are q Teichmüller representatives, as specified by the q roots in $\mathbb{Z}_q$ satisfying $\omega^q = \omega$ given by 0 and the $(q-1)$ roots of unity in $\mathbb{Z}_q$. So in other words, we can also consider the lift of our analysis for $\mathbb{F}_q$ to the ring of Witt vectors $\mathbb{Q}_q$. Summarizing, in the totally unramified case, one has $\mathbb{Q}_q$ as specified by adjoining a primitive $(p^n - 1)$-th root of unity, and one has a π-adic expansion with $\pi = p$.

As another example, in the totally ramified case we can write $\mathbb{Q}_q = \mathbb{Q}_p(p^{1/n})$. This in turn means that we should expect a π-adic expansion with $\pi = p^{1/n}$ a primitive root of p in the unramified case. For more general (possibly ramified) field extensions, there is still a π-adic expansion available, and π is then referred to as a "uniformizer". See [214] for further discussion. For further pedagogical discussion on these points geared towards physicists, see for example references [12, 13]. For some additional discussion on ramification for algebraic number fields and local fields, see Appendix S.

Proceeding in this way, one can also speak of the separable algebraic closure of $\mathbb{Q}_p$, denoted as $\overline{\mathbb{Q}}_p$. We comment that this space is not metrically complete, and adding the points of closure takes us to the space commonly referred to as $\mathbb{C}_p$.[57]

This illustrates a feature we have already encountered in our discussion of finite fields: The notion of "dimensionality" itself is somewhat more subtle compared with the setting of real spacetimes and target spaces. A common line of attack is to view each field extension of $\mathbb{Q}_p$ as adding a full dimension, but as we have already mentioned in our discussion of finite fields, it is sometimes more appropriate to view such field extensions as filling in additional points in a suitable "analytic continuation" of the space.

Now, from the perspective of our previous discussion where we focused on rational morphisms:

$$\phi : X \dashrightarrow Y, \tag{16.10}$$

we see that we can view the path integral in the p-adic context as associated with varieties X and Y defined over a p-adic field. More precisely, once we fix $N = p^a$, we can consider morphisms involving the reduction mod $N = p^a$, i.e.:

$$\phi_{(N)} : X_{(N)} \dashrightarrow Y_{(N)}. \tag{16.11}$$

That this can make sense follows from the fact that we are free to reduce the ring of Witt vectors mod p^a to a ring $\mathbb{W}_{(N)}$ and we are free to consider morphisms over schemes defined over this ground ring. Given this set of morphisms, we can attempt to lift these back to rational morphisms $X \dashrightarrow Y$ defined over our p-adic field. Proceeding in this way, we obtain a regulated notion of a p-adic path integral, as specified by how large we take the exponent a appearing in $N = p^a$.

The formal procedure for defining correlation functions works much as we have already specified in the context of morphisms of schemes defined over finite fields. In particular, we note that the operators of interest will again be given by various $\mathbb{C}^\times$ valued characters. For example, for a morphism to the affine line:

$$\phi : X \dashrightarrow \mathbb{A}^1(\mathbb{Q}_p), \tag{16.12}$$

we can consider the character:

$$\chi : \mathbb{Q}_p \to \mathbb{C}^\times, \tag{16.13}$$

$$t \mapsto \exp(2\pi i\{t\}), \tag{16.14}$$

where $\{t\}$ just means that we drop terms mod p. In terms of our previous discussion involving characters over finite fields, we can emphasize the mod p^a nature of our operators

[57] More generally, given a field K with a norm, it is customary to write $\overline{K}$ to denote a separable algebraic closure, and $\mathbb{C}_K$ to denote its metric completion.

by considering operators such as:

$$\mathcal{O}(t) = \exp\left(\frac{2\pi i}{p^a}\left\{\phi(t)\right\}_{\mathbb{Z}_p}\right), \tag{16.15}$$

namely we only consider the portion of the morphism valued in $\mathbb{Z}_p$. With the eventual aim of passing to a smooth p-adic limit, it seems clear that we can use either presentation of the character map.

Now, carrying things out in this way, we also see that the path integral as specified by morphisms reduced mod p^a comes with a natural notion of ordering which is complementary to the one we discussed for finite fields. Recall that in the finite fields case, we face the subtle point that because "everything is periodic," there is at best only a local notion of time ordering available, and this itself depends on choosing a particular generator for the additive group $(\mathbb{F}_p, +)$.[58]

In the p-adic context, we see that there is another way to partially order elements, as specified by the p-adic norm. Observe that for $t \in \mathbb{Q}_p$, we can write any non-zero element as $t = p^m u$ for some integer m and $|u|_p = 1$ of unit norm. This is very akin to what we have in radial quantization of a 2D conformal field theory, where the radial direction of $\mathbb{C}^{\times}$ serves as a notion of time, with transverse circles serving as the spatial direction. This notion of p-adic ordering is thus quite natural. Indeed, observe that for each p-adic slice (as specified by a power of p^m), we get the unit norm elements. So, we can evaluate at each shell sequentially. Let us also note that this is different from the ordering one would get from the Archimedean completion of the rationals $\mathbb{Q}$ inside $\mathbb{R}$. In that sense, there is an order of limits issue which affects how we evaluate our path integral over morphisms. See figure 8 for a depiction of this notion of "radial ordering".

In fact, a helpful feature of working over the p-adics is that we can then also speak of an emergent topology in the large a limit. In the case of characteristic p varieties, we already saw hints of this emergent structure in our discussion of crystalline cohomology. Here, we see it appearing again, albeit in a somewhat different guise. Let us also note that this also provides a more refined topology than both the Zariski topology and the étale topology that are implicit in our earlier treatment of characteristic p spaces. Returning to our very brief discussion of symmetric bilinear forms defined on $T^*X \otimes T^*X$, we see that the corresponding p-adic expansion:[59]

$$h_{\mu\nu} = \sum_i h_{\mu\nu}^{(i)} p^i, \tag{16.16}$$

also means that there is a suitable notion of a metric for such schemes.

As an amusing application, consider points in the 2D "spacetime" obtained from $\mathbb{Q}_p \times \mathbb{Q}_p$

[58]Recall, however, that we do have a notion of "fast and slow" modes as specified by the degree of a morphism. This in turn provides a notion of past and future.

[59]Compared with earlier, we now label the spacetime indices by Greek rather than Latin indices since here a refers to the exponent of a prime power.

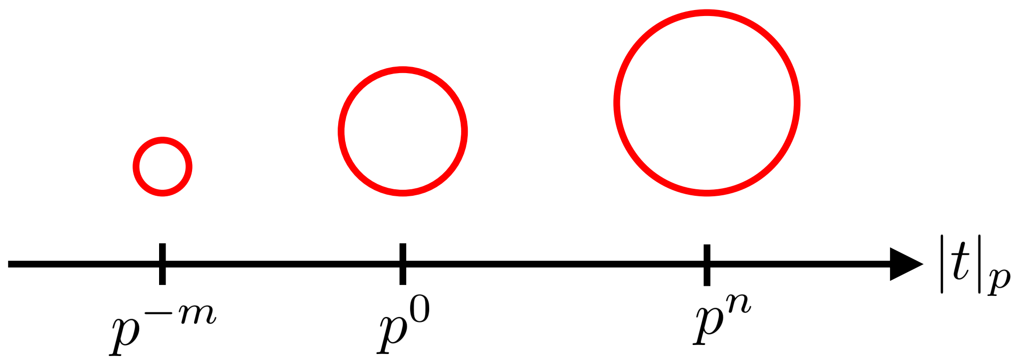

Figure 8: Depiction of radial quantization for $\mathbb{Q}_p$. This provides a notion of "time ordering" for evaluating correlation functions generated by the path integral for morphisms between p-adic varieties.

with $h_{\mu\nu}$ specifying the standard symmetric bilinear form of Minkowski space, namely we can set:

$$h_{\mu\nu}x^\mu x^\nu = (x^0)^2 - (x^1)^2. \tag{16.17}$$

If we restrict to coordinates valued in the rational numbers, we can speak of timelike values $h_{\mu\nu}x^\mu x^\nu > 0$ and spacelike values $h_{\mu\nu}x^\mu x^\nu$, as well as the lightcone $h_{\mu\nu}x^\mu x^\nu = 0$. In the extension to $\mathbb{Q}_p$ where there is no complete ordering, only the notion of the lightcone persists. Another comment here is that in the reduction modulo p^a, we see that a lightcone can take the form:

$$x^0 = x^1 + \alpha p^a, \tag{16.18}$$

which makes sense in any characteristic. So in other words, a single lightcone defined in characteristic p^a breaks up into several disjoint lines inside of $\mathbb{Q} \times \mathbb{Q}$. Consider the large a limit. In the real topology these lines get further and further away from one another, but in the p-adic topology these lines get closer and closer together!

Of course, the p-adic numbers are also rather far removed from our usual notions of metric and distance, at least as far as they are applied in many continuum physical problems. We now argue that in the large p^a limit, this sort of structure also naturally appears. To see why, we note that in evaluating our correlation functions, we actually make implicit reference to the metric on $\mathbb{C}^\times$. This follows simply from the fact that our action principle is really formulated in terms of additive characters of the given ring, namely we have:

$$\exp\left(\frac{2\pi i}{N} S\right) \in S^1 \subset \mathbb{C}^\times. \tag{16.19}$$

Convergence with respect to the metric on $\mathbb{C}^\times$ is of course a very different notion from that specified by the p-adic norm, and gives rise to a completely different sort of topology and notion of "proximity." In the context of our quantum theory, however, we see that actual probabilities / expectation values as computed by the path integral still implicitly make reference to this real metric structure. This raises an important subtlety: if we blindly take the entire infinite series in the p-adic expansion, then we will often produce numbers in $\mathbb{Z}_p$ which are no longer integers. Even so, the notion of a map to $\mathbb{C}^\times$ still makes sense because when we reduce modulo p^a, we get back an integer and the corresponding character is well-defined.

Another helpful comment here is that Mahler's theorem [215] tells us that for any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$, we can obtain an arbitrarily good approximation using a convergent sequence of polynomials $f_n \in \mathbb{Q}_p[x]$. In this sense, the restriction to polynomials we have been considering earlier is actually not much of a restriction at all, at least in the p-adic setting!

Evaluating on a given physical field configuration, the action can still converge in either topology, it just depends on how we take the large N limit. One might argue that for evaluating correlation functions the actual quantity of interest is:

$$\frac{1}{p^a} S = S_a + \frac{1}{p} S_{a-1} + \dots + \frac{1}{p^a} S_0, \tag{16.20}$$

with the leading term set by S_a. In this case, convergence is best thought of in terms of the usual real numbers. We can pass between these two expansions through the formal mapping:

$$\mathbb{Q}_p \leftrightarrow \mathbb{R} \tag{16.21}$$

$$\hbar = \frac{p^a}{2\pi} \leftrightarrow \hbar = 1 \tag{16.22}$$

$$S_j \leftrightarrow S_{a-j} \,. \tag{16.23}$$

To get a character map which converges in $\mathbb{C}^\times$, we need to truncate the p-adic expansion so that $p^{-a} S$ remains small in the real topology. This means that for a fixed value of $N = p^a$ we would need to truncate to terms of degree p^{a-1} or lower:

$$t \sim \sum_{j=0}^{a-1} t_j p^j \text{ and } S \sim \sum_{j=0}^{a-1} S_j p^j. \tag{16.24}$$

We take this to mean that as we pass to the extremely quantum regime where $\hbar \to \infty$, we actually recover a semblance of standard quantum fields. Note also that the "classical limit" corresponds to holding the expansion degree fixed at some maximal $j_{\max}$ and sending $a \to \infty$.

So, depending on how we take our large N limit, we can approach either the p-adic or real topology. If we take the limit at the level of the action, then we pass to the p-adic topology whereas if we take the limit in the space of characters valued in $\mathbb{C}^{\times}$, then we instead pass to the real topology. In the latter case where we use the truncated p-adic expansion of equation (16.16), this also provides us with a metric on our spacetime. Let us note that in the lattice approximation discussed in section 3 as well as the worked example in Appendix A, there is a sense in which we are still referencing the standard real topology by performing "nearest neighbor differences." In such situations, the passage back to the continuum is the standard one. In the more abstract setting based on polynomials, more care as warranted, but the above procedure shows how to accommodate this situation as well.

Our plan in the remainder of this section will be to study some of the structures we can expect to recover in the case where we first take an order of limits where our physical field configurations are associated with morphisms between p-adic varieties. This is the "purest" notion of lifting our characteristic p considerations to a full p-adic expansion. That being said, we will also aim to see how our considerations fit with other notions of physical and mathematical p-adic structures, but we defer this to sections 18, 19 and 20. A related comment is that the proper treatment of differential equations requires us to specify some additional topological structure, as would come from a suitable analytification of the underlying p-adic spaces, a topic we will motivate from a physical perspective in section 20. So long as we are willing to tolerate various formal manipulations (as is customary in physics anyway), then the ordering of topics as presented here should not cause much of an issue.

Our first aim will be to show that our characteristic p starting point provides us with a way to specify an action principle. At least in suitable neighborhoods, we can formulate our field configurations in terms of power series in local variables. That also means that we can take derivatives of these power series, much as we would in working over $\mathbb{R}$ and $\mathbb{C}$. In fact, precisely because such formal power series expansions provide a general method for solving many differential equations, we can essentially *borrow* the corresponding solutions to these differential equations over the real and complex numbers, but now working in the p-adic setting. Aside from the fact that we are here being somewhat glib about the underlying analytic structure associated with our differential equations, we also face the fact that the radius of convergence for these power series is often different as we vary the ground field. As a canonical example, consider the exponential power series:

$$\exp(t) = \sum_{n\geq 0} \frac{t^n}{n!}. \tag{16.25}$$

For $t \in \mathbb{C}$, this converges on $|t| < \infty$, but for $t \in \mathbb{C}_p$, the radius of convergence is $p^{-1/(p-1)} < 1$, namely the series converges for $|t|_p < p^{-1/(p-1)} < 1$ (see Appendix Q for a brief review). Another prominent example which figures in the study of local systems with p-adic monodromy is the logarithm function, with power series expansion:

$$\log(1-t) = \sum_{n\geq 0} -\frac{t^n}{n}, \tag{16.26}$$

which has radius convergence of one, namely the series converges for $|t| < 1$, much as in the real and complex setting. Conversely, there are power series which we can define which have a large p-adic radius of convergence there, but with a rather small radius of convergence over the real and complex numbers.[60]

Our second aim will be to illustrate that we can make sense of a quantum theory in this setting. In particular, we discuss the interpretation of the Schrödinger equation, as well as wave functions, and how to extract probabilities valued in the real numbers. The main stumbling block here is that we will need to give an interpretation of quantities such as the Hamiltonian operator when they have p-adic valued eigenvalues rather than real valued eigenvalues. Since we have already given a quantum mechanical interpretation for the Hilbert space of states associated with physics in the characteristic p setting, there is a lifting procedure we can adopt to obtain a corresponding Hilbert space of states in the p-adic setting. This means, for example, that the wave functions of our system will still be complex valued, but that the field configurations themselves will be p-adic valued.

16.1 Action Principles

The first issue we face in trying to specify a field theory with p-adic valued fields is how to make sense of classical mechanics and quantum mechanics. Our approach in previous sections has mainly centered on the structure of the action, so we begin by revisiting the interpretation of this quantity in our setting. The main idea is that when we consider the character $\exp(iS[\phi]/\hbar)$ with $\hbar = p^a/2\pi$, we can consider a formal sequence where we increase the exponent a. In this sense, each individual action can be evaluated using the same finite sums we have been using in the characteristic p setting. In fact, if we retain this definition, then our physical fields will, in the simplest setting have coefficients in $\mathbb{Z}_p$ rather than $\mathbb{Q}_p$. If, however, we absorb the powers of $\hbar$ into our definition of the fields, then we can consider the closely related character $\exp(2\pi i\{S[\phi]\})$, but where now ϕ can have p-adic coefficients. Here, we have included the brackets "{" and "}" to serve as a reminder that the action is still valued in the p-adics, but that we are then applying a character map to pass over to

[60] As an extreme example, consider the series defined by:

$$f(t) = \sum_{n\geq 0} n!\, t^n. \tag{16.27}$$

Using the same estimates presented in Appendix Q, we have that the p-adic radius of convergence is $p^{1/(p-1)}$. Over the real and complex numbers, however, the only value of t which leads to convergence is the trivial case of $t = 0$.

$\mathbb{C}^\times$.

The reason we have belabored this point is that since we are assuming p-adic valued physical fields, it is a bit awkward to directly work with quantities such as p-adic integrals, which would return real numbers for the measures of sets. In a sense, we are still introducing a measure, but one which is induced from working with the characters valued on $\mathbb{C}^\times$. Let us also remark that there is a notion of residue integral as developed by Coleman [216] and Berkovich [217].

With these comments in place, we can now proceed to write down Lagrangians, much as we would in the real setting. As an example, consider morphisms from the affine line to the affine line, as specified by maps $\phi : \mathbb{A}^1(\mathbb{Q}_p) \to \mathbb{A}^1(\mathbb{Q}_p)$. We can work either with polynomials of bounded degree, namely elements of $\mathbb{Q}_p[t]$, or as is customary in the physical setting, we can view this as implicitly specifying a power series for $\mathbb{Q}_p[[t]]$. We can also extend to rational functions provided we specify prescribed boundary conditions at the locations of the poles. In any case, the main point is that we have a variable t, and so we are free to take "ordinary derivatives" of these expressions. Given such a $\phi(t)$, it then makes sense to write down Lagrangians such as:

$$L[\phi] = \frac{1}{2}(\partial_t \phi)^2 - V(\phi), \tag{16.28}$$

just as we always would. The Euler-Lagrange equation simply specifies a differential equation for $\phi(t)$, and makes sense in this setting as well. By the same token we can also introducing a conjugate momentum $\pi(t)$, and consequently it also makes sense to define a Hamiltonian such as:

$$H[\phi, \pi] = \frac{\pi^2}{2} + V(\phi), \tag{16.29}$$

and develop Hamilton's equations for motion through the corresponding phase space. By design, none of this is very different from the standard setting, all we are doing is interpreting these expressions as having p-adic coefficients. All of this extends in the standard way to higher-dimensional p-adic spacetimes, so we leave this implicit.

There is also not much difference in how we set up the explicit path integration provided we remember to always reduce mod p^a and only later take $a \to \infty$. We shall not attempt to determine whether this limit always exists, but will simply assume it does and explore the consequences. As far as correlation functions go, the natural quantities to consider are again closely tied to $\mathbb{C}^\times$ valued characters such as $\exp(i\phi(t_1)/\hbar)$, in the obvious notation.

Before proceeding to a discussion of the quantum case, it is already interesting to study the special field configurations which produce a stationary complex phase. In the Archimedean setting, such stationary phase contributions are interpreted as "classical" configurations because the ones for which the complex phases of quantum mechanics can all add up constructively for a macroscopic object. By abuse of terminology, we can again refer to these as classical configurations, though the fact that we are working p-adically means that there is an inherently quantum nature to our discussion. Putting aside such concerns, we observe that the principle least of action will produce a p-adic differential equation for $\phi(t)$. To illustrate, consider the case of the harmonic oscillator, with Lagrangian:

$$L = \frac{1}{2}(\partial_t \phi)^2 - \frac{\Omega^2}{2}\phi^2, \tag{16.30}$$

where $\Omega \in \mathbb{Q}_p$ is some p-adic number which in the real setting we would interpret as a characteristic frequency. The Euler-Lagrange equations describe configurations of stationary action, and amount to a $\phi_{\text{osc}}(t)$ which satisfies the p-adic differential equation:

$$\partial_t^2 \phi(t) = -\Omega^2 \phi(t). \tag{16.31}$$

The general solutions take the form:

$$\phi_{\text{osc}}(t) = A_+ \exp(i\Omega t) + A_- \exp(-i\Omega t), \tag{16.32}$$

which we interpret as a power series in the variable t. Here, $i = \sqrt{-1} \in \mathbb{Q}_p(\sqrt{-1})$, and $A_\pm \in \mathbb{Q}_p(\sqrt{-1})$ such that $\phi(t)$ is invariant under $\text{Gal}(\mathbb{Q}_p(\sqrt{-1})/\mathbb{Q}_p)$, in the obvious way. These expressions converge provided we restrict to $|\Omega t|_p < p^{-1/(p-1)}$. See also [36, 218] for some related discussion of the p-adic harmonic oscillator.

Recall that in the real setting, one would get a very similar set of solutions, but Ω would have the interpretation as setting the period of oscillation, namely $T_{\text{period}} = 2\pi/\Omega$. Here, such a notion is more problematic, because the analog of $2\pi i$ in the p-adic setting is rather subtle. The main issue is that there is no clear notion of a non-constant "periodic function" in the p-adic valued setting. Indeed, observe that for any putative $f : \mathbb{Q}_p \to \mathbb{Q}_p$, periodicity would mean $f(t + p^m T) = f(t)$, but this in turn forces f to be constant since $p^m T$ has decreasing p-adic norm as m increases. We also observe that in that setting, the domain of t is not restricted at all.

Our analysis of fermonic systems and supersymmetry also naturally extends to the p-adic setting, since we essentially formulated our entire analysis using in terms of polynomials of arbitrary degree anyway, and we can apply a standard lifting from characteristic p to the p-adics. Indeed, as we already remarked in our discussion of crystalline and rigid cohomology, the supersymmetric quantum mechanics that we introduced implicitly makes reference to such a lifting in the first place.

16.2 Aside on Moduli Spaces

Now, in the standard physical setting, there is a well-known interplay between the vacua of certain supersymmetric field theories and the associated moduli of the target space. This is especially prominent in the case where the field theory is a non-linear sigma model with target space a Calabi-Yau variety. When available, it can prove convenient to characterize this non-linear sigma model as the infrared fixed point of a gauged linear sigma model (of the sort briefly discussed in section 14). Computations of quantities such as the period integrals can then be recast in terms of correlation functions in the corresponding field theory.

One might ask whether we should expect a similar correspondence to hold if we simply switch to characteristic p or the p-adics. An immediate objection is that the notion of a "period integral" itself becomes somewhat more challenging to define in this setting. For example, in the case of a Calabi-Yau n-fold with n odd, we would need to integrate the holomorphic n-form over a real odd-dimensional cycle. The situation becomes even more problematic since we would seem to need a notion of "contour integration" which may or may not persist.

On the other hand, one of the governing features of these sorts of moduli problems is the existence of a Picard-Fuchs differential operator $\mathcal{L}_{PF}$ which acts on the periods Π via the equation:

$$\mathcal{L}_{PF}\Pi = 0. \tag{16.33}$$

The main point we wish to emphasize is that because our analysis has been largely algebraic, the *existence* of this sort of Picard-Fuchs differential equation also makes sense in the p-adic setting. So, we can at least formally definition a notion of moduli and period integrals as specified by the Picard-Fuchs differential equation which they satisfy.

In fact, historically, at least, building examples of "well-behaved" p-adic differential equations often has a strongly geometric character of this sort. This in turn has a remarkable correspondence with counting points on geometries in characteristic p! For some discussion of the associated "Dwork theory" in various contexts, see for example references [157–159, 219, 160, 81, 220, 83–85]. From our present perspective, this is also rather natural to expect because the process of counting points on a characteristic p variety is closely tied to the cohomology theory specified by our supercharge(s) Q.

16.3 Quantum Considerations

The analysis of the previous subsection was classical in nature in the sense that it involved specifying an action principle, and then extracting various p-adic differential equations from a principle of least action. This also extends to other geometric quantities such as the moduli space of a target space, where there is again a differential equation governing the quantities of interest (such as the Picard-Fuchs equations for periods).

Here we ask whether our notion of path integral provides us with a sensible notion of a quantum theory. For some complementary perspectives on p-adic quantum mechanics, see for example [36, 221]. To a certain extent, we can just mimic what we have already done in the even more extreme case of working over a finite field. Recall that in that setting, we introduced a basis of states as specified by morphisms, and then constructed the free vector space over $\mathbb{C}$. In this setting, the phase factor of the path integral $\exp(iS/\hbar)$ is a character map. Nothing much changes in the p-adic setting if we set $\hbar = p^a/2\pi$ with $a \to \infty$. Similar considerations apply to the construction of operators: So long as we confine our analysis to correlation functions involving characters built from physical fields such as $\exp(if(\phi)/\hbar)$ with f some polynomial in the physical field, it seems clear that we can at least formally compute correlation functions.

On the other hand, given the fact that we now have access to additional structure such as p-adic differential equations, we might ask whether we can also demand more, such as a p-adic Schrödinger equation. For this to make sense, we can formally think of a "wave function" as a power series with p-adic coefficients. This would of course obscure the Born rule, but it at least illustrates that at the level of spectral theory, we can speak of a formal space with linear operators which act on it. What is somewhat more awkward is that refining this structure to include a Hermitian pairing (or even a norm) with desired physical properties is more suited to working over $\mathbb{C}$; this is simply because the quadratic extension $\mathbb{Q}_p(\sqrt{-1})$ does indeed have a notion of "complex conjugation," but it naturally leaves open the question about how to treat vectors over the field $\mathbb{C}_p$. On the other hand, it is not clear that we need to make sense of such an operation since if we deal with just character valued maps, we already have the standard complex conjugation operation available from working on $\mathbb{C}^\times$.

Since, however, our procedure is mainly algebraic in nature, we can proceed by considering a basis of formal functions constructed locally as power series in $\mathbb{C}_p[[x]]$. Then, we can introduce linear operators such as the position and momentum operators $\mathcal{O}_{\text{pos}}$ and $\mathcal{O}_{\text{mom}}$ in terms of their action on these:[61]

$$\widehat{\mathcal{O}}_{\text{pos}} f(x) = x f(x) \tag{16.34}$$

$$\widehat{\mathcal{O}}_{\text{mom}} f(x) = \frac{1}{i} \partial_x f(x), \tag{16.35}$$

where $i = \sqrt{-1} \in \mathbb{C}_p$. In the p-adic setting, the significance of the factor of $1/i$ is more obscure because as we already mentioned, we do not have a direct notion of inner product directly on $\mathbb{C}_p$, but rather must appeal to a character map construction. One can then, for example, construct a p-adic Hamiltonian operator such as:

$$\widehat{H} = -\alpha \partial_x^2 + \beta V(x) \tag{16.36}$$

for some $\alpha, \beta \in \mathbb{Q}_p$, and study the corresponding eigenvalue problem for the p-adic differential equation:

$$\widehat{H} f(x) = E f(x) \tag{16.37}$$

[61] The present treatment clearly favors a position basis, and one might ask whether we can work in terms of the conjugate momentum basis instead. We can of course, define a related power series expansion in the momentum variables involving elements of $\mathbb{C}_p[[k]]$, but to complete the circle of ideas, we would need an analytic notion of a "Fourier transform," something which is more awkward to come by in the algebraic setting. We can, however, perform such a Fourier transform by working in terms of the $\mathbb{C}^\times$ valued characters of operators, so we expect such a relation to hold even here. It would be interesting to spell out such a correspondence in more detail, but such concerns will not play much of a role in what follows.

for $E \in \mathbb{C}_p$. Of course, we can generalize beyond just position and momentum variables to include quantities such as spin and angular momentum.

Now, in the Archimedean setting the Hamiltonian is the generator for time translations, and so it is natural to ask whether this also works in the present case. One's first inclination might be to consider a purely p-adic time evolution by introducing a Schrödinger equation such as $\widehat{H} f(x,t) = \gamma \partial_t f(x,t)$. The main difficulty compared with the Archimedean situation is that setting $\gamma = i\hbar$ is not particularly canonical since we have already mentioned the difficulties with specifying a purely p-adic notion of Hermitian conjugation.

At a practical level this sort of notion is not particularly necessary to make sense of time evolution. Again taking our cue from the finite field case, we assume that we our linear operator $\widehat{H}$ is diagonal. In that case, it makes sense to construct a character map for each individual eigenvalue, namely $\exp(iE/\hbar) \in \mathbb{C}^{\times}$ for each E a p-adic eigenvalue. For a rational morphism $\phi_E : X \dashrightarrow Y$ locally expressed in terms of a p-adic power series, it is also clear that the physical state $|\phi_E : X \dashrightarrow Y\rangle$ is an eigenstate of $\exp(i\widehat{H}/\hbar)$ with eigenvalue $\exp(iE/\hbar)$. Then, for a collection of such morphisms, we can even build linear combinations of these energy eigenstates. This provides us with a notion of unitary time evolution on more general physical states. A final generalization amounts to asking about what happens when $\widehat{H}$ is presented as a matrix which is not diagonal. Here, we make the assumption that all entries in $\widehat{H}$ are given by elements in $\mathbb{C}_p$, but that all eigenvalues are p-adic valued. In this case, we adopt the practical procedure of working modulo p^a. Then, we can always replace $\widehat{H}$ by a representative with coefficients in the ring of integers for $\overline{\mathbb{Q}}_p$, and which we truncate at order truncated at order p^a, which we denote as $\{\widehat{H}\}_{p^a}$. Since that matrix has entries which also make sense over $\mathbb{C}$, the evaluation of the character in this case is well-defined. This shows that the notion of time evolution does make sense in the p-adic setting. Observe also that in the regulated case where all eigenvalues of $\widehat{H}$ are assumed to be in $\mathbb{Z}_p$ and we set $\hbar = p^a/2\pi$, the match to Archimedean notions of time provides us with a "minimal time step" as captured by $t_{\min} = 2\pi/p^a \in \mathbb{R}$.

17 State Counting and Period Integrals

As an intriguing application of these considerations, we will also illustrate how the structure of certain Picard-Fuchs equations prevalent in the study of certain supersymmetric systems naturally appears in this setting. The main idea is that arithmetic moduli space problems can be analyzed in terms of a corresponding p-adic differential equation. This enters in a very concrete way in a number of physical settings, especially in the study of supersymmetric vacua with eight or more real supercharges. Celebrated examples include the analysis of 4D $\mathcal{N} = 2$ quantum field theories using Seiberg-Witten theory [222, 223], as well as their counterparts involving compactifications of type II string theory on a Calabi-Yau threefold. So, at least at a formal level, we should expect there to be a natural link between the corresponding mathematical objects appearing in string compactification and their arithmetic counterparts [79–85]. Of course, in matching to physical considerations we ought to demand more, and we expect the considerations spelled out earlier to provide such a mode of analysis.

Roughly speaking, we ask the following physically motivated question. We know that for 4D $\mathcal{N} = 2$ supersymmetric vacua there is a Coulomb branch of moduli space with flat coordinate(s) u. Once we couple to gravity, we expect there to be a maximum field range we can entertain before the effective field theory breaks down. Working in Planck units so that u is dimensionless, we have:

$$|u| \leq |u_{\max}|, \tag{17.1}$$

Where here, we have presented the relation in the case of a one-dimensional Coulomb branch. It is also natural to posit that there is an infrared cutoff, which we interpret as a minimal step size to the field range allowed in our model. The assumption of arithmetic discretization amounts to the condition that both the complex phase and magnitude of u are discretized. Specializing to the case of of real values[62] for u, we assume:

$$\frac{u_{\max}}{u_{\min}} = N \in \mathbb{Z}, \tag{17.2}$$

and we shall interpret this in the same way already outlined in section 3, namely we work in terms of an effective $\hbar = N/2\pi$. For more general complex values of u, it seems natural to restrict to the case where each u is actually an algebraic number, i.e., belongs to a finite field extension of $\mathbb{Q}$.

Having motivated our approach to discretization, we now ask about the profile of our theory in the special case where $N = p^a$ with a taken very large. It is here that we enter the realm of a p-adic geometry for the Seiberg-Witten curve, and its reduction on the residue field to a geometry over a finite field. Our plan in the remainder of this section will be to study the resulting period integrals in this setting, as well as the potential relations between BPS state counting and the Zeta function of the curve. To this end, we first recall some

[62] As a brief aside, one reason for restricting to real values is to demand time-reversal invariance in the 4D theory, much as in reference [224].

basic elements of Dwork theory for the Legendre family of elliptic curves, and then turn to an application and generalizations of these considerations in the context of Seiberg-Witten theory.

17.1 Periods of an Elliptic Curve

To make our discussion a bit more concrete, let us consider the explicit example associated with the Legendre family of elliptic curves:

$$y^2 = x(x-1)(x-u), \tag{17.3}$$

with $u \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$. A classic question in arithmetic geometry is to determine the number of $\mathbb{F}_p$ rational points as we vary u. This in turn can be related to the Picard-Fuchs differential equation associated with this moduli problem [157]. We now briefly review this following for example reference [225] and then present a few physically motivated generalizations.

We begin by interpreting the elliptic curve over $\mathbb{C}$ where there are well-known expressions for the associated period integrals. These involve integrals of the meromorphic one-form dx/y as given by:

$$\omega(u) = \frac{dx}{\sqrt{x(x-1)(x-u)}}. \tag{17.4}$$

We can obtain the Picard-Fuchs differential equation by also computing further derivatives with respect to u:[63]

$$\partial_u \omega = \frac{1}{2} \frac{dx}{\sqrt{x(x-1)(x-u)^3}} \tag{17.5}$$

$$\partial_u^2 \omega = \frac{3}{4} \frac{dx}{\sqrt{x(x-1)(x-u)^5}}. \tag{17.6}$$

Then, one can explicitly verify that a specific linear combination of these derivatives can be expressed as a total derivative:

$$u(u-1)\partial_u^2 \omega + (2u-1)\partial_u \omega + \frac{1}{4}\omega = -\frac{1}{2} d\left(\frac{\sqrt{x(x-1)(x-u)}}{(x-u)^2}\right), \tag{17.7}$$

so the corresponding period Π satisfies:

$$u(u-1)\partial_u^2 \Pi + (2\lambda - 1)\partial_u \Pi + \frac{1}{4}\Pi = 0, \tag{17.8}$$

which is solved by the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; u)$ (i.e., ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; u)$). More gen-

[63]Strictly speaking we should perform this computation using the Gauss-Manin connection, but because u is a flat coordinate, the present treatment suffices.

erally, recall that the hypergeometric functions $F(a, b; c; u)$ satisfy the differential equation:

$$u(u-1)\partial_u^2 F + ((a+b+1)u - c)\partial_u F + abF = 0, \tag{17.9}$$

which in turn can be expressed in terms of a power series:

$$F(a, b; c; u) = \sum_{m \geq 0} \frac{(a)_m (b)_m}{(c)_m} \frac{1}{m!} u^m, \tag{17.10}$$

where in the above, we have:

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(m)} = \frac{a(a+1)...(a+n-1)}{.} \tag{17.11}$$

The appearance of such solutions to differential equations is rather ubiquitous in mathematical physics, and readily extends to a variety of settings. For example, in reference [226], the case of the Bessel function $J_0(u)$ and the p-adic interpretation of its differential equation is also considered in some detail (see also [227]).

17.2 BPS States and Seiberg-Witten Curves

Such period integrals also play a prominent role in a number of theories with eight real supercharges, including those which appear in Seiberg-Witten theory. In fact, the original presentation of a Seiberg-Witten curve given in [223] involved an elliptic curve in the Legendre family (up to an unimportant shift in coordinates), and this can also be generalized in numerous ways (see e.g., [222]). From our present perspective, what is important in Seiberg-Witten theory is that we specify a genus g curve with marked points, along with a meromorphic one-form λ_{SW}. Integration along the A- and B-cycles of the curve produce a set of period integrals, denoted as:

$$a_i^D = \int\limits_{B^i} \lambda_{SW} \text{ and } a^j = \int\limits_{A_j} \lambda_{SW}, \tag{17.12}$$

which depend on a set of flat coordinates for the moduli space, u^i. Moreover, the period matrix of the curve is encoded in the matrix:

$$\tau_{ij} = \frac{\partial a_i^D}{\partial a^j}. \tag{17.13}$$

All of these period integrals are subject to Picard-Fuchs differential equations. See for example [228, 229] for some examples of this sort. The important point for us is that at least in the region of large complex structure, these differential equations have explicit power series presentations with rational coefficients, and this in turn means that they also exist

p-adically.

Compared with the complex analytic case, the continuation of these solutions to other regions of moduli space is more challenging. This is one of the essential features in the study of p-adic differential equations, namely that there exists a suitable notion of "Frobenius structure" which enables one to extend these power series solutions to a larger radius of convergence. Roughly speaking, one first considers the reduction modulo p, and the corresponding Frobenius action on the characteristic p geometry. Then, a suitable lift back to a p-adic space provides a filtration on the differential modules of the geometry. This refinement can then be tracked, and provides a way to specify the profiles of these solutions to the differential equations.

Another quite important element in the physical theory is the location of massless states, as obtained by varying the moduli of the Seiberg-Witten curve. If one were to integrate out such states, one would find a singular effective action. Such singular behavior is also reflected in the geometry of the Seiberg-Witten curve. For example, in the Legendre family (written in Seiberg-Witten's coordinate system) $y^2 = (x-1)(x+1)(x-u)$, the singularities at $u = +1$ and $u = -1$ are associated with states with magnetic charge becoming light, while the singularity at $u = \infty$ is associated with the limit of large W-boson mass. In the process of passing around such a singularity, the periods a_i^D and a^j undergo monodromy, being acted upon by an element from $Sp(2g, \mathbb{Z})$.[64]

Does the notion of "monodromy" have a p-adic counterpart? To a large extent, the key point for us is that there is a corresponding monodromy action in both the complex analytic setting and the p-adic setting, a point emphasized for example in [230]; In the complex analytic setting this involves limiting mixed Hodge structures, while in the p-adic case there is again a suitable notion of weights. A quite abstract and general account can be found in chapters 20 and 21 of reference [231], as well as references therein. The main idea is to frame in purely algebraic data the singular structure of solutions to p-adic differential equations. In this algebraic setting, we can then observe that there is a notion of monodromy which acts on appropriate cohomology groups of the ambient geometry, and its reduction mod p (i.e., reduction to the residue field).

The structure of monodromy follows basically the same contours as what was already outlined in our discussion of the winding modes and the étale fundamental group. To set the stage, we first provide a more formal treatment of the local monodromy group action in the complex analytic setting, following the discussion in, for example [230, 232]. Suppose we are considering a family of elliptic curves E_u such that there is a singular point at $u = u_*$. We can construct a corresponding disk Δ^*, and consider the fundamental group $\pi_1(\Delta^*, u) \simeq \mathbb{Z}$, where u is now associated with a generic point in the family. The fundamental group specifies

[64] We remark that in many physically relevant situations, including that associated with the Legendre family of of curves) the isogeny class of the curve is quite important so one is limited to a finite index subgroup of $Sp(2g, \mathbb{Z})$.

for us a monodromy group action which acts on the cohomology of the elliptic curve:

$$\pi_1(\Delta^*, u) : H^1(E_u, \mathbb{Q}) \to H^1(E_u, \mathbb{Q}), \tag{17.14}$$

and we label the corresponding "monodromy generator" as T. This is a quasi-unipotent operator, and we can also associate to it a nilpotent monodromy operator $\mathbf{N} = \log(1+T)$.[65]

In the arithmetic setting there is a close analog of the fundamdental group provided by the inertia group of a field extension, and this extends to varieties over non-Archimedean and finite fields as well. To illustrate, consider K a non-Archimedean field (in our case this is just a finite extension of $\mathbb{Q}_p$), and a field extension L/K. Then, we can also form the residue fields by quotienting by the ring of integers $\mathcal{O}_L$ and $\mathcal{O}_K$, respectively. Call these residue fields κ_L and κ_K. Then, we have the short exact sequence:

$$1 \to I_{L/K} \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(\kappa_L/\kappa_K) \to 1, \tag{17.15}$$

and we refer to $I_{L/K}$ as the "inertia group". To see the connection with the étale fundamental group, observe that if we take $L = \overline{K}$ the algebraic closure of K, then $\mathrm{Gal}(\kappa_L/\kappa_K) \simeq \widehat{\mathbb{Z}}$. Now, from general properties of ℓ-adic cohomology theory, we also have that $\mathrm{Gal}(L/K)$ acts on $H^1(E_u, \mathbb{Z}_\ell)$, and as such, so too does the inertia group. The generator of the inertia group serves to specify the action of "monodromy" in the p-adic setting. Importantly, there is a generalization of these notions to the case of rigid cohomology simply based on the *algebraic* structure of the corresponding p-adic differential equations for the solutions to the Picard-Fuchs differential equations. As such, we see a quite strong analogy between the physical and arithmetic structures present in both settings. In Appendix T we present a first approximation of these notions by tracking the analog of monodromy actions on the ℓ-adic cohomology of an elliptic curve.

As an additional remark, we note that we have now seen the appearance of two natural actions on the rigid cohomology groups of a variety defined over a finite field $\mathbb{F}_q$; one is associated with the action of the Frobenius map $F : H^i \to H^i$, while the other is associated with the nilpotent monodromy transformation $\mathbf{N} : H^i \to H^i$. One can also establish that these two operations satisfy a sort of "braid relation" (see e.g., [230, 232]):

$$\mathbf{N} \circ F = qF \circ \mathbf{N}. \tag{17.16}$$

The existence of such arithmetic structures is actually somewhat surprising, and hints at the existence of a deeper physical interpretation of the associated rigid and ℓ-adic cohomology groups. The natural context for these issues to crop up in Seiberg-Witten theory is in the structure of the theory once we attempt to couple to gravity, and its relation to the field range of the Coulomb branch parameter u. Following up on our general philosophical remarks in

[65] Recall that a nilpotent operator $\mathbf{N}$ is one such that $\mathbf{N}^k = 0$ for some $k \in \mathbb{Z}_{>0}$. A quasi-unipotent operator T is one such that T^m has eigenvalues equal to one for some $m \in \mathbb{Z}_{>0}$.

section 3, we expect to be able to approximate the value of u by an algebraic number, and as such, we can view it as having a π-adic expansion (π being some possibly non-trivial root of p):

$$u = \sum_i u_i \pi^i, \tag{17.17}$$

in the obvious notation. Now, the case of large field range in the Archimedean setting is sometimes associated with the case of *small* p-adic norm. So, the p-adic extension of Seiberg-Witten theory provides us with at least partial access to the structure of the theory at large field range, as we move closer to the Planck scale. The important point for us is that the notion of a cohomology group, and in particular the notion of monodromy around a singular point still makes sense.

Given our previous interpretation of rigid cohomology groups for curves over a finite field, it is quite tempting to view $H^1_{\text{rig}}(E_u, \mathcal{L})$ as a Hilbert space of states for the particle which is becoming massless. Here, $\mathcal{L}$ refers to a line bundle over the elliptic curve. We shall also be somewhat cavalier with both the specific coefficient ring as well as cohomology theory (be it rigid or ℓ-adic cohomology) since we expect the "generic case" to make sense physically.

To illustrate, we sketch how we expect this to work in the case of a one-parameter family of Seiberg-Witten curves, as parameterized by some Coulomb branch parameter modulus u. We also neglect the contributions from mass parameters, which can also be included as additional moduli (i.e., by working with a punctured curve with additional decoration at the punctures). So, consider the BPS formula for the central charge of electric charge n_{elec} and magnetic charge n_{mag}:[66]

$$Z_{n_{\text{elec}}, n_{\text{mag}}}(u) = n_{\text{elec}} a(u) - n_{\text{mag}} a^D(u), \tag{17.18}$$

where, as already mentioned, we have neglected any contributions from mass parameters. Now, the central charge is itself a period integral, and as such, it is governed by a Picard-Fuchs differential equation.

Our periods a and a^D are to be viewed as solutions to a Picard-Fuchs differential equation satisfied by a suitable meromorphic one-form (the Seiberg-Witten differential), as captured by a section of $\Omega^1(\mathcal{L})$, where $\mathcal{L}$ is a line bundle which depends on the model in question. For example, in the case of the Legendre family (pure $SU(2)$ gauge theory, but where we have shifted the origin of moduli space from $u = 0$.) considered by Seiberg and Witten in reference [222], $\lambda_{SW} \sim (x - u)dx/y$, the differential is a section of the space of meromorphic $(1,0)$-forms with vanishing residue. So, in this case we would just set $\mathcal{L} = \mathcal{O}$, the structure sheaf of the curve. From all that we know about Seiberg-Witten theory, we can also *predict* the quasiunipotent monodromy matrix associated with the rigid and ℓ-adic cohomology groups in "passing around" the singularities in the u-plane: All of these are associated with

[66] In our conventions, the central charge is related to the mass as $M^2 = 2|Z|^2$.

a Kodaira fiber of type I_2, so in our conventions the monodromy action is:

$$A^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \text{ with } A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{17.19}$$

In Appendix T we provide an explicit example illustrating this monodromic structure.

Now, by assumption, we have that a particular state of fixed electric and magnetic charge is becoming light at the point $u = 0$. Call the central charge for this case $Z(u)$. One way to package this change in BPS masses under this Frobenius structure is through the ratio:[67]

$$\xi(u) \equiv (-1)^{(p-1)/2} \frac{Z(u^p)}{Z(u)}. \tag{17.20}$$

On general physical grounds, approaching the massless point signals the approach to a phase transition, and as such, we should expect some sort of singular behavior [233, 234].

To make this more concrete, we introduce a generalized Zeta function over $\mathbb{F}_q$, as obtained by tracking the action of the Frobenius structure on each $H^i(E_u, \mathcal{L})$, which we denote by:

$$F^{(i)} : H^i(E_u, \mathcal{L}) \to H^i(E_u, \mathcal{L}). \tag{17.21}$$

Then, we can form a characteristic polynomial:[68]

$$P_i(z) = \det(\mathbb{I} - zF^{(i)}), \tag{17.22}$$

and the Zeta function:

$$Z_{C_u,q}(z) = \frac{P_1(z)}{P_0(z)P_2(z)}. \tag{17.23}$$

This generalizes to higher dimensions by taking the product over all P_i with odd i in the numerator, and all P_i with even i in the denominator. Compared with the standard "Zeta function for a curve," the main alteration is to allow an arbitrary line bundle. For further review on the Zeta function for a sheaf, see for example reference [156].

Quite remarkably, these physical expectations are borne out by the arithmetic structure of the Picard-Fuchs differential equation! Along these lines, we consider the same Seiberg-Witten curve, but now over the finite field $\mathbb{F}_q$ with modulus v. Let $u \in \mathbb{Z}_q$ denote the Teichmüller representative of v. Then, we can form the product:

$$\Xi \equiv \xi(u) ... \xi(F^{q-1}(u)), \tag{17.24}$$

[67] Our definition of ξ is the inverse of the one typically considered in the math literature. Our reason for choosing this convention has to do with the physical interpretation of ξ in this case.

[68] Our convention follows that in [156]. Observe that the natural range for the index i is from $i = 0, 1, 2$ whereas the curve is "one-dimensional". In general, the cohomology theories of this type are non-trivial up to $2D$ with D the dimension of the variety.

where $F(u) = u^p$. The main case which has been analyzed in the literature is the special situation where $\mathcal{L} = \mathcal{O}$, in which case Ξ turns out to be a zero of the Zeta function $\zeta_{E_u,q}(z)$, precisely what we expect based on physical considerations.[69]

Another simple class of examples are given by the non-compact elliptically fibered K3 surfaces with constant j-function, which are sometimes referred to as H_0, H_1, H_2, D_4, E_6, E_7, and E_8 (see e.g., [235] and references therein). In each of these cases, the Seiberg-Witten differential actually descends from integrating along the u-direction of the holomorphic two-form of the elliptic K3:

$$\Omega_{\mathrm{K3}} = \frac{dx}{y} \wedge du. \tag{17.25}$$

Given all that we said previously, it is natural to expect a similar relation to hold for more general choices of line bundles, and the associated monodromic structure. For example, we can consider more general kinds of singular points in moduli space for the parameter u. Indeed, there is a rich story involving the spectrum of particles which are becoming massless, and the corresponding monodromy type. Examples of this sort include the rank one superconformal field theories H_0, H_1, H_2, D_4, E_6, E_7 and E_8. In the case where we keep the flavor symmetry mass parameters switched off, these curves are given, for non-zero u by (see e.g., [235] and references therein):

$$E_8 : y^2 = x^3 + u^5 \tag{17.26}$$

$$E_7 : y^2 = x^3 + u^3 x \tag{17.27}$$

$$E_6 : y^2 = x^3 + u^4 \tag{17.28}$$

$$D_4 : y^2 = x^3 + \alpha u^2 x + u^3 \tag{17.29}$$

$$H_2 : y^2 = x^3 + u^2 \tag{17.30}$$

$$H_1 : y^2 = x^3 + ux \tag{17.31}$$

$$H_0 : y^2 = x^3 + u. \tag{17.32}$$

The monodromy type associated with each singularity at $u = 0$ is as follows:

$$E_N : A^{N-1}BC^2 \quad (\text{for } N = 6, 7, 8) \tag{17.33}$$

$$D_4 : A^4 BC \tag{17.34}$$

$$H_N : A^{N+1}C \qquad (\text{for } N = 0, 1, 2) \tag{17.35}$$

[69]There is an unfortunate clash of notation between the central charge Z_{central} and the notation for Zeta function we have been using. To avoid confusion we have changed our notation for the Zeta function of the arithmetic curve.

where we have introduced the explicit elements of $SL(2,\mathbb{Z})$:[70]

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}. \tag{17.36}$$

Said differently, the physical intuition here is that the existence of a Hilbert space of states in the p-adic setting would appear to require a specific monodromy group action as well. It would be interesting to directly confirm this physical prediction.

We can also generalize this discussion to account for the presence of mass deformations. Returning to the theories $H_0, H_1, H_2, D_4, E_6, E_7, E_8$, these can all be presented as a deformation of the original Weierstrass model:

$$y^2 = x^3 + f(u; m_1, ..., m_r)x + g(u; m_1, ..., m_r), \tag{17.37}$$

where the $f(u; m_1, ..., m_r)$ and $g(u; m_1, ..., m_r)$ are constructed from the Coulomb branch parameter u, as well as Casimir invariants for each of the corresponding Lie algebras. Here, r indicates the rank of the Lie algebra. Explicit expressions for the f's and g's, including the corresponding Seiberg-Witten differentials for the various cases can be found in various places, including, e.g., reference [235] and references therein.

Finally, let us also mention that another fruitful way to construct examples of Seiberg-Witten geometries is via the spectral cover construction of a Hitchin system on a genus g curve with marked points, as explained in [239,240]. In that setting, the mapping class group of the underlying curve provides us with the analog of the "duality group" action. We have already mentioned that there are characteristic p versions of the Hitchin system available, and analagous statements hold in the p-adic setting as well.

[70]Our conventions follow those given in reference [236]. For a physics centered introduction on these structures, as well as additional connections to the literature (stated in a different convention), see references [237, 238].

18 Holographic Structures

In this section we discuss the sense in which physics on the p-adics is intrinsically holographic. Our discussion is reminiscent of that given in [11–13], but an important distinction is that since our focus is on morphisms defined over the ring of Witt vectors for characteristic p varieties, we do not directly deal with real valued physical fields. We can, however, ask what happens in such situations, and we argue that there is a natural extension of our considerations to that setting in section 19.

To begin, recall that in section 7 we discussed a sense in which scale entangelement in characteristic p systems can be viewed as building up a bulk tree-like space. This becomes especially sharp in the case of p-adic systems.

Indeed, one of the intriguing elements advocated in references [11–13] is that there is a sense in which the p-adic numbers are intrinsically holographic. This can already be seen by just examining the p-adic expansion of any element $t \in \mathbb{Q}_p$ with p-adic norm $|t|_p = p^{-m}$

$$t = p^m \sum_{i=0}^{\infty} u_i p^i, \tag{18.1}$$

with $u_0 \neq 0$, and where for technical reasons which will only be apparent later, we specify the u_i via Teichmüller representatives. The important feature for us is that we can speak of the p-adics with unit norm as specifed by:

$$\mathbb{U}_p = \left\{ u \in \mathbb{Q}_p \quad \text{such that} \quad |u|_p = 1 \right\}, \tag{18.2}$$

and so we can also write for any $t \in \mathbb{Q}_p$, $t = p^m u$ for some $m \in \mathbb{Z}$ and some $u \in \mathbb{U}_p$. From these considerations it follows that the entire set of non-zero p-adic numbers can be written as a disjoint union of "shells":

$$\mathbb{Q}_p^{\times} = \bigsqcup_m p^m \mathbb{U}_p, \tag{18.3}$$

in the obvious notation.

At the most basic level, this provides a hint of holography, because the norm of the number serves as an overall scale. More precisely, we can think of the p possible coefficients t_i in the expansion:

$$t = \sum_i t_i p^i, \tag{18.4}$$

as specifying a node on a tree. This node is then attached to p possible nodes as associated with the coefficient u_{i+1}, and so on off to infinite values of the degree i. What we have just described is the structure of the Bruhat-Tits tree:

$$\mathbb{T}_p \equiv PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p). \tag{18.5}$$

A more formal way to state this same structure is to consider the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ with homogeneous coordinates $[x, y]$. It is well known that the projective line admits an action by $PGL_2(\mathbb{Q}_p)$ just given by transformations of the form:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x + \beta \\ \gamma y + \delta \end{bmatrix}. \tag{18.6}$$

The space of two-dimensional lattices defined as $\mathbb{Z}_p$-modules also admits a natural $PGL_2(\mathbb{Q}_p)$ action, and the Bruhat-Tits tree amounts to the space of equivalence classes $PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$. A helpful reference is [241] (see also [40]), and for an account written for physicists see references [12, 13].

There is clearly a sense in which this tree-like structure is building up a "bulk dual" to the geometry specified by $\mathbb{P}^1(\mathbb{Q}_p)$. In making appeals to the standard AdS/CFT correspondence, one should of course tread carefully because all of this discussion is independent of any particular dynamics (i.e., a choice of a particular CFT on the boundary). But, it is nonetheless suggestive.

Continuing along this route, we see that for each $t \in \mathbb{Q}_p$, we can specify a path in the tree $\mathbb{T}_p$ which begins at a particular node v_t, and extends out to infinity, the "conformal boundary" of $\mathbb{T}_p$. Call this path γ_t. On the other hand, given a node $v \in \mathbb{T}_p$, we also see that it casts a shadow consisting of all points in $\mathbb{Q}_p$ centered at a particular unit u_0:

$$\text{Shadow}(v) = \left\{ t \in \mathbb{Q}_p \quad \text{such that } t = p^m \sum_{i=0}^{\infty} u_i p^i \quad \text{and} \quad u_0 \text{ specified by } v \right\}. \tag{18.7}$$

The discussion extends to the projective line, so we have constructed two canonical maps:

$$\text{point of } \mathbb{P}^1(\mathbb{Q}_p) \to \text{Bulk to Boundary Path in } \mathbb{T}_p \tag{18.8}$$

$$\text{point of } \mathbb{T}_p \to \text{Shadow}(v), \tag{18.9}$$

which is quite reminiscent of various holographic intuitions.

As explained in [12, 13], this discussion also extends to field extensions of $\mathbb{Q}_p$. Perhaps the simplest case is where we work with the completely unramified extension $\mathbb{Q}_q$ (which also has a standard p-adic expansion). In this case, the Bruhat-Tits tree is given by:

$$\mathbb{T}_q \equiv PGL_2(\mathbb{Q}_q)/PGL_2(\mathbb{Z}_q), \tag{18.10}$$

and the uniformizer $\pi = p$, the only change being that now we have $q+1$ branches emanating out from each node (one from the "past" and q towards the "future"). In reference [12,13] this was interpreted as a "higher-dimensional" generalization because it involves a field extension above $\mathbb{Q}_p$. From the perspective of the present note, it still exhibits much of the flavor of a one-dimensional space, since the boundary space is just the projective line $\mathbb{P}^1(\mathbb{Q}_q)$. See

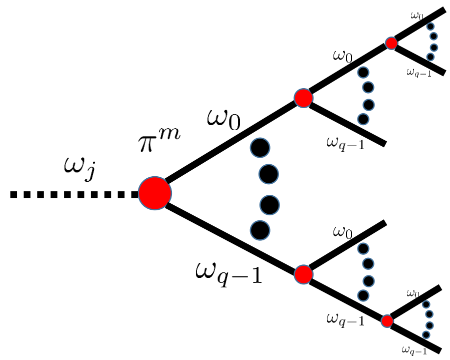

Figure 9: Depiction of the Bruhat-Tits tree for $\mathbb{Q}_q = \mathbb{Q}_{p^n}$, a degree n totally unramified extension of $\mathbb{Q}_p$. We can list elements of $\mathbb{Q}_q$ in terms of a π-adic expansion where π is the "uniformizer" of the field extension for $\mathbb{Q}_q$ over $\mathbb{Q}_p$, and the coefficients are Teichmüller representatives, namely each coefficient satisfies the equation $\omega^q = \omega$. Note that in the case $q = p$, we have $\pi = p$. We label each solution as ω_j for $j = 0, ..., q-1$. Each coefficient is thus listed by q different possibilities, and the resulting sequence of Teichmüller representatives fills out a sequence in a tree where each vertex attaches to $q+1$ vertices.

figure 9 for a depiction of the Bruhat-Tits tree of $\mathbb{Q}_q$.

We faced the same issue on specifying the "dimensionality" of our spacetime and target space in the characteristic p setting, and in that sense it appears to be helpful in viewing $\mathbb{Q}_q$, its algebraic closure $\overline{\mathbb{Q}}_q$, and the metric completion of the algebraic closure $\mathbb{C}_q$ as filling in points as something which in the real setting we would view as "two-dimensional." Nomenclature aside, we can also see a natural generalization to higher-dimensional projective spaces, as dictated by $\mathbb{P}^n(\mathbb{Q}_q)$. In this case, we are really dealing with the more general structure of a building, as specified by the quotient space $PGL_{n+1}(\mathbb{Q}_q)/PGL_{n+1}(\mathbb{Z}_q)$. For further discussion, see e.g., [242].

We can also extend this to more general field extensions $L/\mathbb{Q}_p$ which may have local ramification. In these cases, we still have a uniformizer π and a corresponding π-adic expansion, but the precise structure of the tree in such cases can potentially be more involved. Even so, we still have a Bruhat-Tits tree:

$$\mathbb{T}_L \equiv PGL_2(L)/PGL_2(\mathcal{O}_L), \tag{18.11}$$

where $\mathcal{O}_L$ denotes the ring of integers for L (i.e., those elements with norm less than or equal to one).

We can extend these considerations even further by specifying a boundary field theory with rational morphisms

$$\phi : X \dashrightarrow Y, \tag{18.12}$$

and then in each affine chart isomorphic to $\mathbb{A}^n$ we can set up a corresponding Bruhat-Tits tree.

At this point, it is natural to ask whether we can build a theory in the bulk, and match the profile of bulk modes to boundary correlators, as in the standard AdS/CFT correspondence [243–245] (see e.g., [246, 247] for reviews). On the boundary, we have already emphasized in other contexts that we are interested in morphisms defined over finite fields and their lifts to the ring of Witt vectors. To keep the discussion less abstract, we ask whether there is any notion of a polynomial $\phi \in \mathbb{Q}_p[u]$ (i.e. the coordinate ring of $\mathbb{A}^1(\mathbb{Q}_p)$) or homogeneous polynomials of fixed degree in $\mathbb{Q}_p[u, v]$ (i.e., sections of bundles for $\mathbb{P}^1(\mathbb{Q}_p)$) which can be lifted to a bulk field for $\mathbb{T}_p$. To fix notation, we again write such a morphism $\phi : X \dashrightarrow Y$, in line with our previous discussions.

A fully local action principle in the bulk $\mathbb{T}_p$ is not obvious to us at the moment, but we can at least point to something which is inherently more non-local.[71] A natural answer is that we should really seek a non-local structure in the bulk, as specified by the partially ordered bulk to boundary paths in $\mathbb{T}_p$, which we denote as $\mathsf{PATH}_\infty(\mathbb{T}_p)$. The partial ordering comes from the requirement that at each step of the path we pass to a term with smaller p-adic norm in the Bruhat-Tits tree.

We can also introduce $\mathsf{PATH}(\mathbb{T}_p)$ for the space of all partially ordered paths which may not necessarily end on the boundary. Observe that for two such paths γ and γ' in $\mathsf{PATH}_\infty(\mathbb{T}_p)$, we can interpret these paths as elements in the ring of Witt vectors for $\mathbb{F}_p$. This is the main advantage of using the Teichmüller representatives in the first place. That also means that there is a notion of addition and multiplication for these paths, and so we can freely pass between such paths, and their corresponding points on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$. So, almost at a tautological level, any action we write for the boundary theory can be extended to an action defined over $\mathsf{PATH}_\infty(\mathbb{T}_p)$. Let us denote this extension of ϕ to such paths by Φ. In the evaluation map, it clearly returns sensible answers for paths in $\mathsf{PATH}_\infty(\mathbb{T}_p)$. This is not fully satisfactory, if only because this does not provide a clear notion of quasi-locality in the bulk.

A remedy is available, because we can consider neighboring paths in the tree $\mathbb{T}_p$, say γ and γ' which differ only in that γ' is obtained from γ by appending one additional node v. We can then speak of another path $\gamma - \gamma'$ which has only one non-zero entry in the ring of Witt vectors. Observe that it makes sense to speak of $\Phi(\gamma - \gamma')$, since our evaluation of such terms really stems from a definition given on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$. See figure 10 for a depiction of one such difference of paths.

[71] We will return to this point after we discuss the connection between our approach and "standard" p-adic physics.

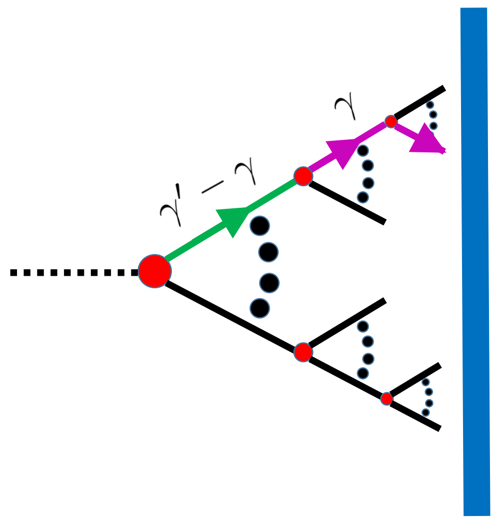

Figure 10: Depiction of two partially ordered paths γ (purple) and γ' (purple and green) in $\mathsf{PATH}_\infty(\mathbb{T}_p)$ in which the partial ordering is dictated by motion towards the "conformal boundary" indicated by the vertical blue line. Each path extends out to the conformal boundary. The difference $\gamma' - \gamma$ (green) is a partially ordered path which does not attach to the boundary. Using such path differences one can construct a quasi-local bulk action from finite differences of such partially ordered paths.

This sort of procedure allows us to specify an evaluation map of the form:

$$\Phi : \mathbb{T}_p \dashrightarrow Y. \tag{18.13}$$

In fact, we could in principle also define a discretized "lattice model" on $\mathbb{T}_p$ by just taking finite differences of Φ at neighboring nodes in the tree. This sort of construction was used in the context of p-adic AdS/CFT (with real valued functions) to propose a bulk action. On the other hand, at many points in this note we have been at pains to point out the limitations of lattice approximations, and have instead appealed to better behaved continuum concepts such as local differentials of morphisms. From this standpoint, it seems much safer (and perhaps conceptually cleaner) to simply view such a lattice action as an approximation for a more non-local structure of the sort we have implicitly defined in terms of partially ordered bulk to boundary paths. We will revisit this issue again in section 20, where we will consider an analytification of p-adic varieties to Berkovich space. In Berkovich space, we again have tree-like structures with derivatives constructed from finite differences on the tree. The difference, however, is that the full space can be viewed as the inverse limit of a family of trees [248], and for this reason, it allows us to bypass some of the standard difficulties with lattice approximations to derivatives.

18.1 Entangled p-adic Numbers

To draw out the parallels with standard holographic considerations, it is helpful to consider the structure of entangled states and their associated scale dependence. Along these lines, recall in section 5 that we introduced the notion of various Hilbert spaces as specified by spatial morphisms $X_s \dashrightarrow Y$. Clearly, these considerations carry over to the p-adic setting. Moreover, because the "timelike" direction X_t comes with a built in p-adic norm, we see that time-ordering now makes more sense (modulo the fact that we have the "radial" direction of a p-adic number to contend with). We also saw in section 7 that a tree-like structure naturally builds up a notion of entanglement across scales.

With this in mind, we now show how the small Hilbert space in the p-adic setting $\mathcal{H}_{\text{small}}(\mathbb{Q}_p)$ is closely related to the big Hilbert space on the affine line $\mathcal{H}_{\text{big}}(\mathbb{A}^1(\mathbb{F}_p))$, and leads to a notion of "number entanglement" similar to the scale entanglement encountered in the finite characteristic setting in section 7. The considerations we present clearly extend to the totally unramified case, where we compare $\mathcal{H}_{\text{small}}(\mathbb{Q}_q)$ and $\mathcal{H}_{\text{big}}(\mathbb{A}^1(\mathbb{F}_q))$. Further generalizations are also available because the case of local ramification in the p-adic setting just amounts in the characteristic p setting to dealing with function fields with ramified extensions. For ease of exposition we focus on the simplest case which still illustrates all the main points.

Consider, then, a p-adic number $t \in \mathbb{Z}_p$, the ring of integers. Observe that there is a formal power series:

$$t = \sum_{m \geq 0} \omega_m p^m, \tag{18.14}$$

where here we use the Teichmüller representatives ω_n. In the small Hilbert space, each such point specifies a state for us, and we can write this instead as a sequence of Teichmüller representatives:

$$|\omega_0, ..., \omega_i, ...\rangle = \left| t = \sum_m \omega_m p^m \in \mathbb{Z}_p \right\rangle \tag{18.15}$$

We also have a p-dimensional qudit Hilbert space $\mathcal{H}_{\text{Teich}}$ which is spanned by the possible Teichmuller representatives $|\omega\rangle$. In this way, we again build up a tensor product structure for $\mathcal{H}_{\text{small}}(\mathbb{Q}_p)$ which is reminiscent of the one already encountered in section 7 for $\mathcal{H}_{\text{big}}(\mathbb{A}^1(\mathbb{F}_p))$. We can first form a Hilbert space consisting of truncation to degree M in the p-adic expansion:

$$\mathcal{H}_{\text{small}}(\mathbb{Z}_p)^{(M)} \simeq \mathcal{H}_{\text{Teich}}^{(m=0)} \otimes ... \otimes \mathcal{H}_{\text{Teich}}^{(m=M)}. \tag{18.16}$$

Then, we can build up the entire small Hilbert space $\mathcal{H}_{\text{small}}(\mathbb{Z}_p)$ from the inverse limit:[72]

$$\mathcal{H}_{\text{small}}(\mathbb{Z}_p) \simeq \varprojlim \mathcal{H}_{\text{small}}(\mathbb{Z}_p)^{(M)}. \tag{18.17}$$

[72] One can of course extend this to the small Hilbert space of $\mathbb{Q}_p$.

Entanglement of states works in essentially the same way as already discussed in section 7.1. As an example, the superposition of $|0\rangle$ and $|1+p\rangle$ specifies a GHZ state:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}|0,0\rangle + \frac{1}{\sqrt{2}}|1,1\rangle, \tag{18.18}$$

in the obvious notation. Continuing in this way, we obtain a notion of entanglement amongst different numbers.

As another suggestive example, fix x a positive integer and define the state given by summing over prime numbers ℓ not exceeding x (in the Archimedean norm):

$$|\Pi(x)\rangle = \frac{1}{\sqrt{\pi(x)}}\sum_{\ell\leq x}|\ell\rangle, \tag{18.19}$$

where $\pi(x)$ denotes the number of primes not exceeding x, and where for each $|\ell\rangle$ we introduce the corresponding p-adic expansion with Teichmüller representatives. Observe that once the value of $|x|_{\mathbb{R}}$ (the Archimedean norm) becomes at least as large as the next prime after p, the state $|\Pi(x)\rangle$ is generically entangled. It would be quite interesting to directly compute this entanglement entropy, and it is likely closely tied to the distribution of primes. Indeed, it is a classic result (the prime number theorem) that the asymptotic distribution of primes satisfies [249, 250]:

$$\lim_{x\to\infty}\pi(x)\times(-x^{-1}\mathrm{log}x^{-1}) = 1, \tag{18.20}$$

where we have suggestively written the second factor as an entropy function.

19 "Standard" p-adic Physics

At this point we would be remiss if we did not mention there has been much recent work on understanding potential physical interpretations for working over the p-adics. The goals of the present approach appear to be somewhat distinct from what has been pursued elsewhere, but our aim will be to establish some basic connections with these complementary (and substantially older as well as more thoroughly investigated) proposals.

Doing so will allow us to make contact with the more "standard" literature on p-adic physics (see e.g., [33–41] for earlier work, as well as references [11–13, 42–64] for more recent work which also includes new applications to holography).

The approach we have advocated so far is somewhat distinct from the "standard" treatment of physical systems defined over p-adic numbers. A very common starting point is to consider a bosonic field, as specified by a map from $\mathbb{Q}_p$ to $\mathbb{R}$, namely:

$$\phi_R : \mathbb{Q}_p \to \mathbb{R}. \tag{19.1}$$

The subscript R serves to remind us that these are real valued functions as opposed to the algebro-geometric morphisms we have been considering up to this point. Real valued functions are somewhat more awkward to handle from our current perspective but it is of course important to see whether we can make contact with the existing literature, and in what way.

In our case, the big Hilbert space of states we have constructed consists of morphisms $|\phi : X \dashrightarrow Y\rangle$. In particular, since our path integral still involves sums over characters taking values in $\mathbb{C}^\times$, we also know that all overlaps of states also take values in the complex numbers. In particular, we can speak of a wave functional $\Psi[\phi(x_s)]$ which depends on the spatial profile of a field. This wave functional again takes values in the complex numbers.

Now, in the context of quantum field theory, we are accustomed to viewing the quantum field as a convenient device for describing multi-particle excitations at different locations in spacetime. From this physical perspective, the existence of the wave functional $\Psi[\phi(x_s)]$ means that we should expect a set of real valued physical fields which capture the same dynamics.

How to construct this basis of fields in practice? To illustrate, we consider again the simplest situation with morphisms from $\mathbb{A}^1(\mathbb{Q}_p) \to \mathbb{A}^1(\mathbb{Q}_p)$, as specified by polynomials $\phi \in \mathbb{Q}_p[x]$. We can build a p-adic valued function by using the evaluation map for this polynomial, and by abuse of notation we write these values as $\phi(t)$ for $t \in \mathbb{Q}_p$. We can produce a $\mathbb{C}^\times$ valued function using the character map:

$$\mathcal{O}(t) = \exp(2\pi i\{\phi(t)\}). \tag{19.2}$$

Taking a logarithm, one can also construct a real valued function which is well-defined up to branch cuts:

$$\phi_R(t) \equiv \frac{1}{2\pi i} \log \mathcal{O}(t) = \frac{1}{2\pi i} \log \exp(2\pi i \{\phi(t)\}), \tag{19.3}$$

i.e., we have converted the evaluation of a p-adic morphism to a real valued function.

In the context of "standard" p-adic physics, one can also construct a notion of an action, including kinetic terms and potential energy terms. The case of a potential energy density involves further composition with real valued functions with p-adic support, and a natural subclass of possibilities involves just having further polynomials in real valued fields:

$$f : \mathbb{Q}_p \to \mathbb{R}. \tag{19.4}$$

The case of kinetic terms requires us to specify some notion of a derivative. From a physical perspective, the main thing we wish to retain is a notion of a momentum eigenstate. More precisely, we seek differential operators $D^{(m)}$ which compose via $D^{(m)} D^{(n)} = D^{(m+n)}$ and which act on characters $\chi_k(t) = \exp(2\pi i \{kt\})$ as $D^{(m)} \chi_k(t) = |k|_p^m \, \chi_k(t)$, i.e., the characters are the p-adic generalization of a plane wave. This can be accomplished using the Vladimirov derivative [36, 38, 37]. A helpful discussion of the Vladimirov derivative is given in Appendix B of reference [13]. Here, we focus on the main elements of these results. For a general real valued function, we have:

$$D^{(m)} f(t) \equiv \frac{1}{\Gamma_p(-m)} \int\limits_{\mathbb{Q}_p} dt' \frac{f(t') - f(t)}{|t' - t|_p^{m+1}}, \tag{19.5}$$

where we have introduced the standard Haar measure for $\mathbb{Q}_p$ such that $\mathbb{Z}_p$ has unit volume. Here, we have also introduced the p-adic Gamma function:

$$\Gamma_p(\alpha) = \int\limits_{\mathbb{Q}_p} dt \ \chi(t) \, |t|_p^{\alpha - 1} = \frac{1 - p^{\alpha - 1}}{1 - p^{-\alpha}}, \tag{19.6}$$

where $\chi(t) = \exp(2\pi i \{t\})$. We shall sometimes use the notation $D = D^{(1)}$ to indicate the special case of a single Vladimirov derivative.

To see that this definition accomplishes the main task, we follow reference [13], and act on a character $\chi_k(t)$. Doing so, we have:

$$D^{(m)} \chi_k(t) = \frac{1}{\Gamma_p(-m)} \int\limits_{\mathbb{Q}_p} dt' \frac{\chi_k(t') - \chi_k(t)}{|t' - t|_p^{m+1}} = \frac{\chi_k(t)}{\Gamma_p(-m)} \int\limits_{\mathbb{Q}_p} dt' \frac{\chi_k(t' - t) - 1}{|t' - t|_p^{m+1}}. \tag{19.7}$$

We can consider the change of coordinates $k(t' - t) = x'$. Then, we get:

$$D^{(m)} \chi_k(t) = |k|_p^m \, \frac{\chi_k(t)}{\Gamma_p(-m)} \int\limits_{\mathbb{Q}_p} dx' \frac{\exp(2\pi i \{x'\}) - 1}{|x'|_p^{m+1}}. \tag{19.8}$$

Evaluation of this integral is somewhat subtle because the denominator and numerator both vanish as $|x'| \to 0$. In references [251, 252] this integral is evaluated. Following reference [251], we can break up the integral as a sum over shells of fixed norm $S_m = \{t \in \mathbb{Q}_p \text{ such that } |t| = p^m\}$. Evaluation of the integral can then be achieved using the value of the integral of a character over the ball $\mathbb{B}_m = \{t \in \mathbb{Q}_p \text{ such that } |t| \leq p^m\}$:

$$\int_{\mathbb{B}_m} dx \ \chi(kx) = \begin{cases} p^m & \text{if} \quad |k|_p \leq p^{-m} \\ 0 & \text{otherwise} \end{cases} . \tag{19.9}$$

The end result is that one now finds:

$$D^{(m)} \chi_k(t) = |k|_p^m \, \chi_k(t), \tag{19.10}$$

as claimed. In principle, then, we can consider a mode expansion of a real valued function f and decompose into Fourier modes such as:

$$\widetilde{f}(k) = \int_{\mathbb{Q}_p} dx \ \chi_k(x) f(x). \tag{19.11}$$

For example, we can form a kinetic term using by acting via:

$$\int_{\mathbb{Q}_p} dx \ D^{(m)} f(x) D^{(n)} f(x) = \int_{\mathbb{Q}_p} dk \ \ |k|_p^{m+n} \, \widetilde{f}(k) \widetilde{f}(-k). \tag{19.12}$$

From our present perspective where we treat the affine line $\mathbb{A}^1(\mathbb{Q}_p)$ as a one-dimensional space, the seemingly natural choice is to take a kinetic term with $m + n = 2$. On the other hand, in the p-adic string theory literature where $\mathbb{Q}_p$ is interpreted as the worldsheet of an open string, it is also natural to consider $m + n = 1$. The reason for this choice is that if we Fourier transform back to position space, then the Green's function is:

$$\langle f(x) f(y) \rangle \sim -\log |x - y|_p \,, \tag{19.13}$$

in accord with what one expects for the Archimedean string [253]. That being said, we also note that for a quadratic extension of $\mathbb{Q}_p$, there is a sense in which we have a two-dimensional space and then producing a logarithmic Green's function would require $m+n = 2$, so it seems reasonable to retain our main thread where we specify a kinetic term by a two derivative, two field term.[73]

It is also interesting to consider the action of the Vladimirov derivative on the operators

[73] There is some debate in the p-adic string theory literature as to the proper way to formulate closed p-adic strings. We also note that even in the open string case, the driving aim appears to be to give a p-adic

indicated in equation (19.2):

$$D^{(m)}\mathcal{O}(t) = \frac{1}{\Gamma_p(-m)}\int\limits_{\mathbb{Q}_p} dt' \frac{\mathcal{O}(t') - \mathcal{O}(t)}{|t'-t|_p^{m+1}} \tag{19.15}$$

$$= \mathcal{O}(t)\frac{1}{\Gamma_p(-m)}\int\limits_{\mathbb{Q}_p} dt' \frac{\exp(2\pi i\{\phi(t'+t) - \phi(t)\}) - 1}{|t'-t|_p^{m+1}}. \tag{19.16}$$

Insofar as the lattice derivative provides an approximation in the small t' approximation, we can write:

$$\phi(t'+t) - \phi(t) = t'\partial\phi(t) + ... \tag{19.17}$$

as an adequate approximation. So, in other words, we get to leading order:

$$D^{(m)}\mathcal{O}(t) = \mathcal{O}(t)\,|\partial\phi(t)|_p^m + ... \tag{19.18}$$

For further discussion on related manipulations, see for example reference [255].

Constructing a kinetic term of the sort we have already proposed now follows from also including $\mathcal{O}^\dagger(t) = \exp(-2\pi i\{\phi(t)\})$ so that in particular we have:

$$D\mathcal{O}^\dagger(t)D\mathcal{O}(t) = |\partial\phi(t)|_p^2 + ... \tag{19.19}$$

So, at least formally, we can pass between the algebro-geometric setup defined in terms of Jacobi sums of characters and the standard construction of p-adic actions.

Similar considerations hold in the construction of a "bulk action" defined on the Bruhat-Tits tree. In reference [12], the graph defined by the Bruhat-Tits tree $\mathbb{T}_p$ was used to specify a lattice model, with each vertex (i.e., node of the graph) of the group associated with a real number $\varphi : \mathrm{Vert}(\mathbb{T}_p) \to \mathbb{R}$. Then, a lattice kinetic term can be achieved by taking nearest neighbor differences, via:

$$\sum_{\langle vw\rangle} (\varphi_v - \varphi_w)^2\,, \tag{19.20}$$

version of the Veneziano amplitude (see [254] for the Archimedean case), through expressions such as:

$$B(\alpha,\beta) \equiv \int\limits_{\mathbb{Q}_p} dx|x|_p^{\alpha-1}|1-x|_p^{\beta-1}. \tag{19.14}$$

To get this sort of structure to appear from "worldsheet" correlators, one can adopt the single derivative action (so that a logarithm is produced). A related comment is to recall the notion of p-adic time ordering we developed earlier where we view $\mathbb{Q}_p$ as specified by elements with a fixed norm, as well as a choice of unit. In that sense, working with $\mathbb{Q}_p$ could be viewed as "two-dimensional" while $\mathbb{Q}_p \times \mathbb{Q}_p$ might then be viewed as "four-dimensional". We can in some sense bypass these issues if we adhere to the notion of dimensionality already advocated in our development of physics over finite fields. From this perspective, one ought not to be biased by intuition derived solely from working over the real numbers. We present a related proposal for formulating non-Archimedean strings in section 20.

where $\langle vw \rangle$ denotes nearest neighboring vertices v and w in $\mathbb{T}_p$. Recall that in our discussion of morphisms supported on paths in the Bruhat-Tits tree that we also encountered p-adic valued functions with support in $\mathsf{PATH}_\infty(\mathbb{T}_p)$, which we then extended to functions valued on the vertices of the graph. Using the character map, we can now take, for $\Phi \in \mathsf{PATH}_\infty(\mathbb{T}_p)$, the character $\exp(2\pi i \{\Phi\})$, and from this, construct a corresponding finite difference. So in this sense, we expect to make contact with the holographic discussions in references [12, 13].

From this point on, we can essentially borrow much of the discussion from the existing literature, but with the provisos that sometimes the notion of spacetime dimension we have discussed earlier is distinct from how it is used in the p-adic physics literature. It would clearly be interesting to further develop these similarities and differences, especially as it pertains to formulating a more general notion of p-adic strings.

With this in mind, let us comment on a few "obvious points" which emphasize more the use of the p-adics. Observe, for example, that we can specify Calabi-Yau spaces over $\mathbb{C}$ by the condition that the canonical bundle is trivial. This of course also makes sense over $\mathbb{C}_p$, so one might ask whether there is a more direct relation between these two geometries. For example, in the case of a complete intersection Calabi-Yau, we can consider specializing the coefficients of the hypersurface equations to have integer coefficients. In this case, we can directly formulate the geometry over either $\mathbb{C}$ or $\mathbb{C}_p$.

In fact, if we restrict ourselves to algebro-geometric structures, even more is possible. This is because there is a non-canonical isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$, and as such, we can freely interchange the ground field, which we can loosely write as a relation such as $X(\mathbb{C}) \simeq X(\mathbb{C}_p)$.[74]

From this perspective, the notion of "proximity" really depends on more refined structures such as the metric. It also means, however, that as far as studying standard questions such as intersection theory of divisors or other numerical invariants (as often appear in the physics literature), we are free to use either ground field.

Another comment is that in passing from p-adic spaces to real (or complex spaces), we can make use of the characters of the p-adics. Let us note here that in the context of for example, the Hitchin system, it is natural to work in terms of the corresponding character variety for the moduli space of solutions.

[74] The relation between the two can be seen as follows (see e.g., reference [161]). Our discussion follows [256] (see also [257]). First, we introduce transcendence bases for $\mathbb{C}$ and $\mathbb{C}_p$, which we write as T and T'. By definition, this means $\mathbb{C} \simeq \overline{\mathbb{Q}(T)}$ and that $\mathbb{C}_p \simeq \overline{\mathbb{Q}(T')}$. Note, however, that as sets, $\mathbb{C}$ and $\mathbb{C}_p$ have the same cardinality, and therefore so do T and T'. Consequently, using the axiom of choice we can conclude that there is a non-canonical isomorphism $\mathbb{Q}(T) \simeq \mathbb{Q}(T')$. Taking the algebraic closures yields $\mathbb{C} \simeq \mathbb{C}_p$. Note that this also means that for distinct primes p and p' we also have $\mathbb{C} \simeq \mathbb{C}_p \simeq \mathbb{C}_{p'}$. From a number-theoretic perspective this is clearly awkward, but at least formally, nothing stops us from carrying out this sort of correspondence.

20 Analytification

In the previous section we saw the emergence of p-adic geometry, at least in the limit where we take $N = p^a$ with a large. Now, as we approach this limit, we can ask whether we expect to recover physics over a p-adic variety, or perhaps some other space. Indeed, in comparing with what happens with physics over the real numbers, we can see a number of important differences. For one, the topology of $\mathbb{Q}_p$ is not path connected, and the space of real valued continuous functions includes locally constant functions. These can perhaps be viewed as important distinctions compared with the setting of physics over the reals, but if our goal is to eventually recover standard physical constructs, we should also aim to see how more refined topological structures can also emerge from this setting. Indeed, one of the main virtues of working with an analytic continuation of real quantities is that one can then in principle leverage structures from complex analysis. Historically this has been an important theme in the development of physical theories, so it would seem worthwhile to further develop it in this setting as well.

Another related issue is that while it is of course appealing[75] to formulate our path integral in terms of discrete sums over characters, one can of course ask whether there is a sensible notion of convergence. A common strategy in the physical setting is to consider the analytic continuation to Euclidean signature, since partition sums in statistical mechanics often have better behaved convergence properties. We have already remarked that there is a formal way to accomplish this even in characteristic p, but it still relies on the formulation in terms of characters for a complete formulation. Provided we can set up a suitable notion of analysis over a p-adic variety, we can hope that both these topological and analytic issues can be dealt with simultaneously.

Our aim in this section will be to address these issues.

Now, at first glance, the "standard" approaches to p-adic physics do not appear to possess such structures. For example, one proposal in the p-adic string theory literature is to treat the genus zero worldsheet for open strings as $\mathbb{Q}_p$, and a suitable quadratic extension of $\mathbb{Q}_p$ as the closed string worldsheet. While the intuition for doing this is that a quadratic extension of $\mathbb{R}$ is just the complex numbers, it seems fair to say that some of the most important features of complex analysis typically used in the study of standard perturbative string theory are wholly absent from such p-adic analogs. To keep the discussion somewhat closer, one might instead consider the algebraic closure of $\mathbb{Q}_p$, denoted as $\overline{\mathbb{Q}}_p$. We note that our discussion of physics in characteristic p naturally makes contact with this field, as obtained from the ring of Witt vectors associated with the algebraic closure $\overline{\mathbb{F}}_p$. Now, it turns out that $\overline{\mathbb{Q}}_p$ is not metrically complete, but there is a standard metric completion which is also algebraically closed, and it is denoted by $\mathbb{C}_p$. More generally, for a local field k we refer to $\mathbb{C}_k$ as the metric completion of its algebraic closure.[76] At first glance, then, $\mathbb{C}_p$ would appear to be a

[75] At least in our opinion.

[76] A comment on notation: In early sections we reserved K for the ground field and L for various extensions.

natural candidate for carrying out a more direct link with standard structures appearing in real physics.

But as is well known, even this space has a rather coarse topology. To name just one issue, $\mathbb{C}_p$ is not path connected. Moreover, when working over $\mathbb{C}_p$, there is apparently no natural notion of analysis in the same spirit as what would occur over the standard complex numbers $\mathbb{C}$. All of these issues make more refined discussions of physics difficult to achieve.

In this section we argue that some of this additional structure is already present in the way that we have been setting up our general framework for path integrals over finite fields. What we shall argue is that in the large a limit, the Hilbert space of states constructed from morphisms between varieties can be supplemented by additional points which "fill in" the topology of the p-adic variety. This procedure is, in the mathematics literature, known as "analytification". Starting from a variety X defined over a non-Archimedean field K which is metrically complete and algebraically closed, there is a corresponding p-adic analytic space X_{an}. We will mainly focus on the simplest case where this procedure produces physically compelling structures, and this is known as the Berkovich space associated to X. One of our aims will be to see how far we can get in recasting our previous considerations in terms of physics defined on Berkovich space. The important point for us is that once this analytification procedure is completed, we get a refined topology on which many manipulations similar to complex analysis can be carried out, but now in the non-Archimedean setting. This in turn allows us to bypass some of the more awkward features of p-adic physics, as well as allowing us to make far closer contact with how we expect physics over the real numbers to eventually emerge. For a depiction of the tree-like structure of Berkovich space, see the depiction provided in figure 11.

The rest of this section is organized as follows. We begin by briefly reviewing some aspects of Tate algebras and their use in defining a rigid analytic space. In particular, we explain how our path integrals are already sensitive to such structures. At some level, this equips a p-adic variety with a refined Grothendieck topology. This still produces a rather coarse topology. We then explain how a further analytification is possible, as in the work of Berkovich. This further procedure also emerges naturally in our setting by considering the physical implications of defining local observables in our setting.

20.1 Topological Refinements

In this section we argue that our path integral formalism naturally motivates working over a rigid analytic space, and its further refinement to a Berkovich space. This is not the place to give a full account of these notions, but we would at least like to sketch how they naturally appear from the considerations we have already set in motion.

We did this in part to avoid confusion with the standard use of k as a momentum variable in the physical setting. Since we will need to discuss various field extensions in the analytic setting, in this section we will aim to be more flexible in our notation choices.

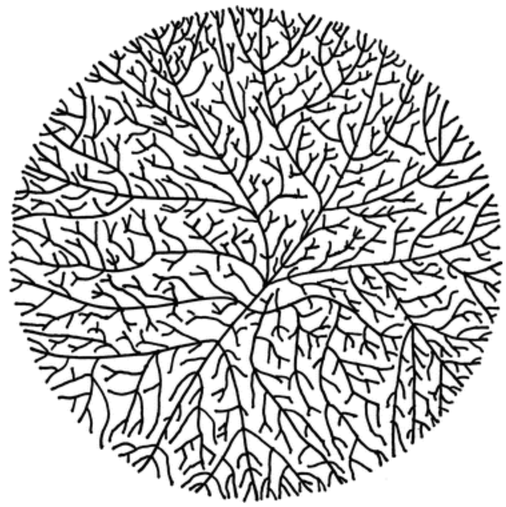

Figure 11: Depiction of the Berkovich projective line reproduced from figure 2 of reference [258]. See also references to Yggdrasil in The Poetic Edda, Völuspá, stanzas 19, 47; Grímnismál, stanzas 35, 44; and Fjölsvinnsmál, stanzas 19, 20.

To set the stage, we first consider the original structure of rigid analytic geometry proposed by Tate [31], and its algebro-geometric characterization given by Raynaud [259]. A helpful account is given in reference [260]. We begin by considering a local non-Archimedean field k with $K = \widehat{\overline{k}}$ the metric completion of the algebraic closure. We denote the n-dimensional unit ball inside k^n by:

$$\mathbb{B}^n = \{(t_1, ..., t_n) \in k^n \quad \text{such that} \quad \max |t_i| \leq 1\}\,. \tag{20.1}$$

The main idea is to consider next power series in $k[[T_1, ..., T_n]]$ with suitable convergence properties on $\mathbb{B}^n$. Writing an element $f \in k[[T_1, ..., T_n]]$ as:[77]

$$f = \sum_{\nu_i \geq 0} a_{\nu_1 ... \nu_n} T_1^{\nu_1} ... T_n^{\nu_n}, \tag{20.2}$$

the condition that f converges on $\mathbb{B}^n$ occurs if and only if $|a_{\nu_1 ... \nu_n}| \to 0$ as $\nu_1 + ... + \nu_n \to \infty$. Observe that the collection of such convergent power series forms an algebra (actually what is known as an affinoid algebra) and we refer to this as T^n. Observe that the norm on k extends to one on T^n because for any $f \in T^n$ we can write the Gauss norm for this power series as:

$$|f| = \max |a_{\nu_1 ... \nu_n}|\,. \tag{20.3}$$

This makes sense because convergence of f on $\mathbb{B}^n$ means that there is indeed a maximal coefficient. The appearance of these sorts of power series can be viewed as a limiting opera-

[77] We have switched to the standard polynomial variables which appear in the literature.

tion as we proceed to the p-adic limit of our path integrals over finite fields. An additional comment is that physically, we often specify local operators in terms of their behavior on a region of the spacetime. This is precisely what we get when we focus on convergent power series on the n-ball $\mathbb{B}^n$.

We can already see some hints that notions from complex analysis have analogs in this setting, since for example we have a maximum modulus principle:

$$|f| = \max_{t \in \mathbb{B}^n} |f(t_1, ..., t_n)|, \tag{20.4}$$

in the obvious notation. Of course, the "nicest situation" is where we work directly with respect to $k = K$ so that no further field extensions need to be considered, but we can proceed more generally, at least for now.

At this point we can proceed as we would normally in algebraic geometry, and construct the points of the polydisk as specified by maximal ideals in T^n. From this starting point, we can consider ideals I and quotients T^n/I. These produce k-Banach algebras, and are referred to as affinoids. These can be viewed as specifying subsets inside the polydisk, and so we can construct a corresponding Grothendieck topology from the corresponding notion of a covering space. Said differently, just from asking for convergent power series, we obtain a first notion of a rigid analytic geometry. At this point, the operations of gluing can be used to build more general notions of gluing spaces, and so we see that a variety X over k can be functorially related to its analytification X_{an}.

There are, however, still some unsatisfactory elements in this construction, because Grothendieck topologies are still somewhat coarse. Physically what we would really like to see is a clear notion of how to connect paths between distinct points in our topology, i.e., we would like to see connected paths such as $\gamma : [0, 1] \to X$, where $[0, 1]$ denotes the standard unit length interval in $\mathbb{R}$. The reason we expect this to be possible is due to the fact that we want to have a notion of time ordering which makes sense in the infinitesimal setting. As we have already explained, there is a sense in which we can fix this issue "by hand" in the context of finite fields, and even p-adic spaces by just imposing a notion of discrete time step. At a technical level, having such path connectedness is also desirable because it gets us much closer to standard physical intuition which has been developed based on methods from complex analysis.

Quite remarkably, it turns out that there is a canonical way to add these additional points which include the construction of Berkovich [32, 261], as well as Huber's more general notion of adic spaces [262]. Again, giving a full acount of these notions would be a rather long digression, but we can at least give a condensed account in Appendix U. For our purposes, the main point is that there is a suitable analytification of a p-adic variety which provides us with a notion of a path connected space.

Rather than continue on in the most abstract setting, let us now specialize to the one-dimensional case given by the projective line $\mathbb{P}^1_{\text{Berk}}$, as well the related spaces specified by the affine line $\mathbb{A}^1_{\text{Berk}}$ and the "upper half space" $\mathbb{H}_{\text{Berk}} = \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(K)$. The main thing we want to emphasize here is that these spaces are path connected, and moreover, can be equipped with a suitable topology which is metrizable, as well as more standard notions from analysis such as a Radon integral and Laplacian. Of particular significance for us is that there is also a well-defined potential theory available on Berkovich spaces, and this makes it possible to give a more satisfactory account of correlation functions in this limit. Then, we can of course consider the suitable truncation of these notions to various p-adic and even finite characteristic spaces.[78]

Along these lines, we note from reference [263] that for $x, y \in \mathbb{A}^1_{\text{Berk}}$, we have a canonical notion of distance between x and y relative to infinity denoted by $\delta(x, y)_\infty$. This can be specified using a formal operation on the tree-like structure associated with the Berkovich projective line, or perhaps more concretely as the limit:

$$\delta(x, y)_\infty = \limsup_{(x_0, y_0) \to (x, y)} |x_0 - y_0| \,, \tag{20.5}$$

where $x_0, y_0 \in K$ and we use the product topology on $\mathbb{A}^1_{\text{Berk}} \times \mathbb{A}^1_{\text{Berk}}$ to define the lim sup operation. In the literature, this is sometimes written as $[x, y]_\infty$ and is known as the Hsia kernel.

There is also a notion of a Laplacian for real valued functions $f : \mathbb{A}^1_{\text{Berk}} \to \mathbb{R}$, as well as a standard notion of Green's function given by:

$$G(x, y) \equiv -\log \delta(x, y)_\infty. \tag{20.6}$$

Owing to the tree-like structure of the space $\mathbb{P}^1_{\text{Berk}}$, this can be evaluated using a nearest neighbor differences formula on the associated graph, but we emphasize that this Laplacian retains more of the desirable analytic features one would want from such a differential operator compared with a crude lattice approximation (as would be specified over the reals). In particular, this Laplacian $\Delta_{\mathbb{P}^1_{\text{Berk}}}$ has the important property that acting on the Green's function yields:

$$\Delta_{\mathbb{P}^1_{\text{Berk}}} G(x, y) = \delta_y - \delta_\infty, \tag{20.7}$$

where in the above, we imagine taking derivatives with respect to the first argument x. Additionally, δ_y and δ_∞ denote Dirac delta functions in the sense that they are concentrated

[78] Indeed, there is a version of these notes in which one could have attempted to entirely reverse the order of the various parts and sections, starting from a formulation over Berkovich spaces fibered over $\operatorname{Spec} \mathbb{Z}$ and then proceeded to various coarse graining operations, namely, first restricting to a single prime of $\operatorname{Spec} \mathbb{Z}$, then proceeding to $\mathbb{C}_p$, then to some finite extension of $\mathbb{Q}_p$ and finally to a suitable residue field such as $\mathbb{F}_q$ or $\mathbb{F}_p$. Our own view is that this would likely have led to an (even more) jarring account, but it would likely be instructive. We leave this as an exercise to the interested reader.

around the points y and ∞, and integrate to one, i.e.:

$$\int\limits_{\mathbb{A}^1_{\text{Berk}}} dx \quad \delta_y = 1. \tag{20.8}$$

In fact, even more is available to us. As found in [264], the analytification of a variety over a local field K can be equipped with a very close analog of the standard (p, q) differential forms which appear prominently in the study of complex analytic varieties![79] The main construction involves notions from tropical geometry, and treating Berkovich's construction as a suitable inverse limit on the associated tropicalizations, so we simply refer the interested reader to the original papers for additional details. Instead, we focus on some of the close parallels available to us in this setting. The main elements are remarkably close to what one typical expects in standard complex analytic treatments involving Dolbeault cohomology. As reviewed in [265], some elements directly carry over from the complex case with a suitable notion of differential form, as specified by Lagerberg's notion of "supercurrents and superdifferentials" (no relation with supersymmetry) given in reference [266].

Let us first briefly recall how superforms work for real varieties, and then we discuss how this extends to p-adic analytic spaces [266]. Our discussion follows the helpful treatment presented in [264] (see also [267]). To begin, we note that for any real manifold, we can introduce a notion of a local (p, q)-differential by forming the tensor product:

$$\mathcal{A}^{p,q}(U) \equiv C^\infty(U) \otimes \bigwedge^p V^* \otimes \bigwedge^q V^*, \tag{20.9}$$

where V denotes a vector space over the reals. Locally, we can then write a (p, q) form as:

$$\omega = \sum_{|I|=p,\ |J|=q} \omega_{IJ}\ dx_I \otimes dx_J, \tag{20.10}$$

in the obvious notation. Said differently, one simply takes two copies of the de Rham complex and forms a bi-grading from these two complexes.

Quite remarkably, these notions can also be extended to p-adic analytic spaces using some methods from tropical geometry. We briefly review some of this in Appendix V. The upshot is that there is a notion of (p, q) real differential forms on X_{an} which we denote as $\mathcal{A}^{p,q}$. The essential point is that the tropicalization map for affine n-space:

$$\text{Trop: } \mathbb{A}^n \to \mathbb{R}^n \tag{20.11}$$

$$(z_1, ..., z_n) \mapsto (-\log|z_1|, ..., -\log|z_n|). \tag{20.12}$$

So, building the suitable bigraded differentials on the image of the tropicalization, there is a

[79] Here, p does not refer to a prime number, but just the degree of a differential form. The clash of notation is unfortunate, but hopefully the context will be clear. Similar comments hold for q.

natural pullback of this from the real setting to the p-adic analytic setting. In other words, we can also speak of (p,q)-forms in the p-adic analytic setting.[80] For a helpful overview to this, as well as related topics, see e.g., [268].

To illustrate the similarities with the complex case, let us briefly recall that for a complex analytic variety Z, we often make use of Dolbeault differential operators ∂ and $\overline{\partial}$ with $d = \partial + \overline{\partial}$, and we can summarize all of this in terms of a triple $(Z, \partial, \overline{\partial})$. In particular, we can label differential forms according to their (p,q) type with respect to holomorphic and anti-holomorphic indices, i.e., we can write complex valued differential forms $\mathcal{A}^k$ via the decomposition:[81]

$$\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}. \tag{20.13}$$

Moreover, there is a natural link between the cohomology of the complex $(\mathcal{A}^{p,\bullet}, \overline{\partial})$ and $H^q(Z, \Omega^p)$.

Turning next to the case of X_{an} the analytification of a variety defined over K, a remarkable result of [264] is that there is a quite similar notion of differential operators d' and d'' such that $d = d' + d''$, and moreover, that there is an analogous triple $(X_{\mathrm{an}}, d', d'')$ with a grading of analytic differential forms $\mathcal{A}^{p,q}_{\mathrm{an}}$ such that:

$$d' : \mathcal{A}^{(p,q)}_{\mathrm{an}} \to \mathcal{A}^{(p+1,q)}_{\mathrm{an}} \tag{20.14}$$

$$d'' : \mathcal{A}^{(p,q)}_{\mathrm{an}} \to \mathcal{A}^{(p,q+1)}_{\mathrm{an}}. \tag{20.15}$$

In fact, even more is available, and there is even an analog of the Poincaré-Lelong formula, namely for a divisor D specified by f a meromorphic function such that $\mathrm{div} f = D$, we have:

$$d'd'' \log |f| = \delta_{\mathrm{div}\ f}. \tag{20.16}$$

This means, in particular, that we can decompose our Laplacian into a composition by d' and d'', much as we would do in the "ordinary" complex setting. There are, of course, some distinctions which show up, since in the p-adic setting the space of locally constant functions is somewhat richer than in the standard complex setting, but other than this, we find it promising that so much of the structure which appears in the study of complex differential geometric objects can be transported over.

This in turn provides a fresh perspective on some of the more conjectural aspects of our discussion of physics in characteristic p presented earlier. While we there aimed to build up various physical structures directly, we can now see that once embedded in the bigger structure of Berkovich spaces, there is a certain "inevitability" (albeit still quite conjectural)

[80] The reason we need to work with an analytification rather than just a variety over $\mathbb{Q}_p$ or $\mathbb{C}_p$ is that we need to have a refined enough topology to properly construct suitable pullbacks. If the topology is too coarse, this can fail.

[81] We are running out of sensible letter choices in the Latin alphabet. The meaning of the notation for the various superscripts k, p, q should be obvious.

to some of these claims.

With this in place, let us now turn to the construction of some physical systems which exhibit some of this additional structure.

20.2 Building Actions

Compared with the situation appearing in "standard" p-adic physics, we see that formulating an action principle in this setting is now rather straightforward. Moreover, this discussion is naturally geared towards what in the real setting we would have referred to as the "Euclidean signature"formulation. Rather than present a complete treatment, we will let the presentation of examples serve the same purpose.

As a first example, consider the theory of a free scalar field $\phi : \mathbb{A}^1_{\mathrm{Berk}} \to \mathbb{R}$. This can be specified by the action:

$$S_{\mathrm{free}}[\phi] = \int\limits_{\mathbb{P}^1_{\mathrm{Berk}}} dx \, \left(-\phi \Delta_{\mathbb{P}^1_{\mathrm{Berk}}} \phi \right). \tag{20.17}$$

On a curve Σ we can also write this using our differential operators d' and d'' on Berkovich space as:

$$S_{\mathrm{free}}[\phi] = \int\limits_{\Sigma} - \phi d' d'' \phi. \tag{20.18}$$

Let us also remark that at this point, we can, by the physicists' standard abuse of notions of limits, set up a path integral formalism, integrating over the space of all such maps. Then, we can evaluate real valued correlators much as we would in a standard 2D field theory defined over the complex projective line:

$$\langle \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n) \rangle = \frac{\int [d\phi] \exp(-S[\phi]) \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n)}{\int [d\phi] \exp(-S[\phi])}, \tag{20.19}$$

where of course we can now entertain more general choices for our action.

Note also that since we have a satisfactory notion of Green's functions available, we can expect to use an analog of the method of images to set up similar actions for the affine line $\mathbb{A}^1_{\mathrm{Berk}}$ as well as the half space $\mathbb{H}_{\mathrm{Berk}}$. Additionally, since we can proceed patch by patch, similar notions clearly extend to general genus g curves initially formulated over $\mathbb{C}_p$ and its analytification.

There are also important differences from the case of working over the complex numbers. For example, if we consider a disk in $\mathbb{C}$, then the boundary is always an S^1. In the Berkovich setting, however, this boundary is often a finite number of points, owing to the tree-like structure of Berkovich space (see e.g., [263] for some helpful examples).

We can also formulate an action principle for single derivative fields, including fermionic degrees of freedom. Indeed, since we have the differential operators d' and d'', the construction of a suitable Dirac operator is readily available. To keep the discussion concrete, let us again focus on the one-dimensional case, with Σ a curve. Then, we can form close analogs of the standard bc and $\beta\gamma$ systems of 2D CFTs / string theory. Along these lines, suppose we have b and c given as meromorphic sections of the line bundles $\mathcal{K}_{\Sigma}^{1/2} \otimes \mathcal{L}$ and $\mathcal{K}_{\Sigma}^{1/2} \otimes \mathcal{L}^{\vee}$ with $\mathcal{L}$ a line bundle, and we have made a choice on the spin structure. Then, we get a well-defined action:

$$S_{bc} = \int_{\Sigma} b d'' c, \tag{20.20}$$

The usual case of the bc system corresponds to setting $\mathcal{L} = \mathcal{K}_{\Sigma}^{3/2}$, while a standard fermion is obtained by setting $\mathcal{L} = \mathcal{O}_{\Sigma}$. The point here is that nowhere do we need to actually demand that we are working over a complex curve. Indeed, everything goes through as expected when Σ is a Berkovich space. We can clearly provide a similar setup for the $\beta\gamma$ system, as well as more general actions constructed via our differential operators.

This is all by design and points to the general feature we want to emphasize. There is little distinction at this point between how a physicist typically *algebraically* manipulates various correlation functions, be it over the complex numbers or the analytification provided by a Berkovich space.

Continuing with our theme of one-dimensional systems (i.e., two "real" dimensions), we also see that we can also introduce a standard notion of a stress energy tensor, much as we would in an ordinary 2D CFT.[82] To illustrate, consider again the real valued scalar $\phi : \Sigma \to \mathbb{R}$. Observe that we can construct a $(1,0)$-form $d'\phi$. So, on the symmetric product $\mathcal{A}^{(1,0)} \otimes \mathcal{A}^{(1,0)}$, we can build a corresponding "holomorphic" stress tensor:

$$\mathcal{T} = d'\phi \otimes d'\phi, \tag{20.21}$$

and we can, in turn, also compute operator correlation functions using the fact that we have the usual logarithmic dependence for the two-point function of free scalars. In particular, this allows us to specify a precise notion of a stress energy central charge $c_{\mathcal{T}}$, just as we would in ordinary conformal field theory. Similar considerations hold for the bc and $\beta\gamma$ systems introduced on Berkovich space.

Another comment is that precisely because we have operators d' and d'', we see that we can also formulate supersymmetric versions of our theory. Indeed, because the usual verification that a Lagrangian is invariant under supersymmetry just involves a sequence of algebraic manipulations, there is again, a functorial sense in which the construction of a supersymmetric theory is automatic. We summarize this by the formal relation between real / complex valued actions and their formal counterparts obtained from analytification:

$$S_{\mathbb{C}} \leftrightarrow S_{X_{\mathrm{an}}}. \tag{20.22}$$

[82]The absence of a stress energy tensor has historically been a major issue in the study of the p-adic string (we thank A. Huang and B. Stoica for emphasizing this point to us). Here, it appears more naturally.

In a certain sense, we already anticipated that this ought to be possible in our construction of characteristic p supersymmetric actions, but in that setting, our emphasis was on target spaces which were also characteristic p varieties. Of course, from our discussion of p-adic field theories in section 16, we have seen that we can go back and forth between the two settings via suitable character maps. At the algebro-geometric level, this is actually not that surprising since there is a formal isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$, which equally well applies to any two fields with transcendental elements. Convergence properties of the path integrals is of course, not guaranteed to work under such a map, but at the level of morphisms between varieties, we see that our discussion allows us to go back and forth.

20.3 Brief Aside on Holography

We previously mentioned that the structure of the p-adics has a rather holographic flavor, something which has been explored in great detail in [11–13], as well as in its potential relations to tensor network models [8–10]. One issue with some of these developments is that while suggestive of a bulk / boundary correspondence, some dictionary entries from the standard AdS/CFT correspondence are not so apparent. In particular, the identification of a "bulk graviton" is challenging to construct in this setting and this in turn makes its role in the study of quantum gravity less immediate. Now, in the standard AdS/CFT correspondence, one considers the dual of the graviton is just the stress energy tensor. Since we have now constructed a p-adic analog of the stress energy tensor (via our superdifferential construction), we can simply start tracking the p-adic expansion of this boundary operator, which in turn tells us the bulk profile for the graviton. Similar considerations clearly hold for higher-dimensional varieties. For example, we can take the product of K-analytic curves and performing the corresponding p-adic expansion. Note also that the p-adic expansion always introduces a single "extra dimension" which again mimics the structure present in Archimedean holography. So, aside from providing us with a way to set up a stress tensor for string worldsheet CFTs, we can apply the same construction more broadly.

20.4 Closed Berkovich Strings

At this point it becomes irresistible to at the very least attempt a formulation of p-adic string theory, but where now the worldsheet is specified on the corresponding Berkovich space rather than directly on a p-adic variety. Doing so, we get to leverage all of the additional structure present in the analytification of our variety.

Before getting to this, let us mention that setting things up in this way allows us to address a somewhat awkward feature of the "standard" accounts of open and closed p-adic string theory, where typically $\mathbb{Q}_p$ or some quadratic extension are considered. This notion of "dimensionality" seems to sacrifice too much of the analytic structure present in the standard physical string, so we can at least hope that the present effort can help us in this regard.

For one thing, we observe that there is a relatively clear notion of what we could mean by evaluating a string amplitude in this setting: One first specifies the usual vertex operators for scalars such as ϕ, evaluates a suitable vertex operator correlation function, and then proceeds to integrate over any worldsheet moduli which cannot be fixed by global conformal killing vectors. All of these notions make sense in the context of Berkovich spaces so it is clear that many notions carry over in a functorial sense. To make complete sense of string theory in this setting would require us to specify a notion of a moduli space of metrics on the Berkovich space, but again, this is something that has at least been considered in some detail in the case of curves (see e.g., [269] and references therein). Indeed, we expect that the moduli space of complex structures of a curve is also a Berkovich space.

The usual notions of string theory also require the appearance of worldsheet fermions, but again, we saw how to implement this in our discussion of the bc and $\beta\gamma$ systems. This also makes it clear that the usual cancelling of the conformal anomaly (i.e., the Weyl anomaly) generated by the bc system and the $\beta\gamma$ system (in the supersymmetric case) directly transports over to the case of Berkovich space. Additionally, this means that for the standard bosonic string, the critical dimension is just 26, and for the superstring (i.e., the case with worldsheet supersymmetry) we again get ten dimensions. Again, none of this is too different from standard string theory.

Observe also that we have available a genus expansion. In the present setting, this follows from the fact that because we have well defined notion of differentials and sheaf cohomology on Berkovich space, as well as differential operators d' and d'', there is a corresponding index theorem which tells us, for example, that on Σ a genus g curve with no marked points and L a line bundle, we have:

$$\mathrm{Ind}(d''_{\mathcal{L}}) = h^0(\Sigma, \mathcal{L}) - h^1(\Sigma, \mathcal{L}) = \int\limits_{\Sigma} ch(\mathcal{L}) Td\,(\Sigma) = \deg \mathcal{L} + (1-g) \tag{20.23}$$

with $ch(\mathcal{L})$ the Chern character, and $Td(\Sigma)$ the Todd class. In particular, for $\mathcal{L} = \mathcal{O}_\Sigma$ the structure sheaf, we have

$$\mathrm{Ind}(d'') = h^0(\Sigma, \mathcal{O}_\Sigma) - h^1(\Sigma, \mathcal{O}_\Sigma) = (1-g), \tag{20.24}$$

so we can as usual, organize our string amplitudes according to a genus expansion, weighted by $\lambda^{-2\mathrm{Ind}(d'')}$, with λ the string coupling.

It is also of interest to try and calculate a scattering amplitude in this setting. To give a sketch of how such a computation can proceed, we observe that from the generalization of our free action (20.17) to the case of multiple ϕ^A:

$$S_{\mathrm{free}}[\phi] = \int\limits_{\mathbb{P}^1_{\mathrm{Berk}}} dx \ \left(-G_{AB}\phi^A \Delta_{\mathbb{P}^1_{\mathrm{Berk}}} \phi^B\right), \tag{20.25}$$

we can evaluate a correlation function between two vertex operators:

$$\left\langle \exp(ik_A\phi^A(x))\exp(il_A\phi^A(y))\right\rangle = \left|\delta\left(x,y\right)_\infty\right|^{k\cdot l}, \tag{20.26}$$

as would be associated with two tachyon vertex operators. Evaluation of string amplitudes proceeds much as in the Archimedean case, namely, introduce closed string vertex operators:

$$V_j = \exp(ik_j\cdot\phi(x_j)). \tag{20.27}$$

Continuing in this vein, we can also see how to set up many of the standard notions from ordinary open and closed string theory, for example we can build vertex operators such as:

$$V^{AB} = d'\phi^A d''\phi^B \exp(ik\cdot\phi), \tag{20.28}$$

as would be associated with the graviton (when A and B are symmetrized). The Virasoro conditions on the momenta lead to the standard conditions on $k^2 = k_A k^A$, since we have the some notion of a stress energy tensor.

So, for example, m-point tachyon scattering amplitudes will involve:

$$\mathfrak{M}^{\text{closed}}_{\text{m-point}} \sim \int\limits_{\left(\mathbb{A}^1_{\text{Berk}}\right)^m} dx_1...dx_m \ \left\langle V_1...V_m\right\rangle, \tag{20.29}$$

where we have omitted various superfluous proportionality factors such as the string coupling and the inverse volume of global "conformal" transformations. In the special case of four-point tachyon scattering amplitudes, there is one "mobile point" $x\in\mathbb{A}^1_{\text{Berk}}$,which we cannot fix using global conformal transformations of $\mathbb{A}^1_{\text{Berk}}$:[83]

$$\mathfrak{M}^{\text{closed}}_{\text{4-point}} \sim \int\limits_{\mathbb{A}^1_{\text{Berk}}} dx \ \left|\delta\left(x,0\right)_\infty\right|^{k_1\cdot k_2}\left|\delta\left(x,1\right)_\infty\right|^{k_2\cdot k_3}, \tag{20.30}$$

which we might view as a Berkovich space analog of the Virasoro-Shapiro four tachyon closed string amplitude (see [270,271] for the Archimedean case). Here we have dropped the overall dependence on the string coupling and for now we suppress the superscipts and subscripts, since the context is clear.

One might also ask how "open string" amplitudes fit into this sort of picture. From the way we have defined the hyperbolic Berkovich space $\mathbb{H}_{\text{Berk}} = \mathbb{P}^1_{\text{Berk}}\backslash\mathbb{P}^1(\mathbb{C}_p)$, it seems natural to view the open string as residing on the Galois orbit invariant subspace of "the boundary" $\mathbb{P}^1(\mathbb{C}_p)$, namely just $\mathbb{P}^1(\mathbb{Q}_p)$. In this sense, evaluation of tree level open string scattering amplitudes is quite similar to what is usually proposed in the p-adic string theory literature. The distinctions start to arise because rather than just considering a quadratic field extension

[83]This group should be viewed as the analytification of $SL(2,\mathbb{C}_p)$.

of $\mathbb{Q}_p$, it seems more natural to consider the limiting behavior of field extensions, and the ensuing analytification in Berkovich space as the "natural" arena for the closed string. For these reasons, we will first focus on the case of our closed string, and only then turn to the case of open strings (precisely the opposite line of development from how p-adic strings are typically studied).

To proceed further, then, we need to evaluate this sort of expression. To carry this out, we view $\mathbb{C}_p$ as the limit obtained from finite field extensions of $\mathbb{Q}_p$. Since $\mathbb{A}^1(\mathbb{C}_p)$ is dense in $\mathbb{A}^1_{\text{Berk}}$, we expect that evaluation of equation (20.30) can be defined through a suitable limit of the form:

$$\mathfrak{M} = \lim_{n\to\infty} M_n, \tag{20.31}$$

with:

$$M_n = \int\limits_{\mathbb{Q}_q} dx \ \ |x|^{k_1\cdot k_2} \, |1-x|^{k_2\cdot k_3} \,, \tag{20.32}$$

in the obvious notation. In Appendix W we carry out this computation, and observe that there is a well-defined limit as $n \to \infty$. The end result is written in terms of a suitable generalization of the Euler Beta function:

$$\mathfrak{B}(a,b) = \int\limits_{\mathbb{A}^1_{\text{Berk}}} dx \ \ |x|^{a-1} \, |1-x|^{b-1} = \frac{1}{1-p^{-a}} \frac{1}{1-p^{-b}} \frac{1}{1-p^{-c}}, \tag{20.33}$$

where we have:

$$\mathfrak{M} = \prod_{x=s,t,u} \frac{1}{1-p^{\alpha(x)}}, \tag{20.34}$$

where s, t, u reference the usual Mandelstam variables and we have, following the conventions in [35], introduced:

$$\alpha\,(x) = 1 + \frac{1}{2}x. \tag{20.35}$$

As an amusing comment, it is also natural to consider the "adelic amplitude" produced from taking the product over all the different primes. In this case, we get:

$$\mathfrak{M}_{\text{adelic}} = \prod_{p\text{ prime}} \mathfrak{M}_p = \prod_{x=s,t,u} \zeta(-\alpha\,(x)), \tag{20.36}$$

where ζ is the celebrated Riemann zeta function. It would be interesting to give a physical interpretation for the zeros of $\mathfrak{M}_{\text{adelic}}$, perhaps in terms of its relation to the "prime at infinity" which is implicitly specified by the condition:

$$\mathfrak{M}_{\text{adelic}}\mathfrak{M}_\infty = 1. \tag{20.37}$$

20.4.1 Current Algebras

The considerations presented above also show that it is possible to specify a notion of a chiral current algebra on this worldsheet, and so in turn to implement a variant of the worldsheet theory for the heterotic string. To see why, it is enough to again exploit the appearance of the operators d' and d''. In particular, we supplement the left-movers by a collection of free fermions, writing an action such as:

$$S_\lambda = \int_\Sigma k_{ab} \lambda^a d'' \lambda^b, \tag{20.38}$$

where the indices a, b run over a set of flavor indices, and k_{ab} is a set of couplings. Here, the λ's transform as a section of $\mathcal{K}_\Sigma^{1/2}$, i.e., as "$(1/2, 0)$" differential forms (in complex geometry terms). From this, we can clearly start following the usual procedure for implementing a worldsheet current algebra, i.e., by considering composite bilinear operators built from the λ's. In particular, from the way we have set up the action principle, the structure of correlation functions for the resulting currents:

$$J^{ab} = \lambda^a \lambda^b, \tag{20.39}$$

will be quite analogous to what happens in the complex setting. This also means that we can introduce a notion of a target space gauge algebra, much as we would in the standard heterotic string. The fact that we also have a notion of the conformal anomaly also suggests that we have quite similar restrictions on the admissible gauge groups (which in the complex setting follow from the condition of modular invariance).

20.5 Open Berkovich Strings

It is also natural to ask whether we can define a notion of open Berkovich strings. Here, we expect to make closer contact with the p-adic string theory literature, where several important structural features of the theory have already been worked out. The main thing we will aim to do is see how to reconcile (if possible) our perspective on closed strings specified on Berkovich space with these older considerations. In particular, we wish to understand the potential ways in which open string structures such as Chan-Paton factors can arise in this setting.

To begin, we note that our discussion of physics over the p-adics has emphasized the "two-dimensional" nature of $\mathbb{Q}_p$, since, in contrast to the real numbers $\mathbb{R}$, there are an infinite number of p-adic numbers with the same norm, whereas in the real setting, there are precisely two for each non-zero number. This is also related to the fact that there is no total ordering on the p-adics, only a partial ordering as specificed by the norm of elements. Now, in the Archimedean string, the open string worldsheet is, at tree level, just a disk, the boundary of which is an $S^1 \simeq \mathbb{P}^1(\mathbb{R})$. Consequently, we have a well-defined notion of local path ordering on the circle, which allows us to introduce Chan-Paton factors, as well as a way to multiply the corresponding matrices, i.e., just by composition of maps.

The absence of total ordering on the p-adics clearly complicates this. In [272], a proposal was also given to rectify the problem which involves dividing up the p-adics into "positive" and "negative" elements, as specified by whether they are in the image of the Norm map $\mathrm{Norm}_{L/\mathbb{Q}_p} : \mathbb{Q}_q \to \mathbb{Q}_p$, with L a quadratic extension. Our present considerations clash with this for a few reasons. First of all, in terms of our formulation based on Berkovich space we have advocated, the extension of this notion eliminates all but 0 and 1 in $\mathbb{Q}_p$ as "positive elements", which is clearly too restrictive. Said differently, why stop at quadratic extensions?[84] An additional concern is to remain somewhat true to the requirement that we can extend our discussion beyond the simplest case of tree level string scattering processes. That in turn means that we require a flexible notion of how we specify a "worldsheet with boundary".

How then to proceed? There are some complementary notions of a tree level worldsheet with boundary that we can reference, and some of this is in accord with expectations from the p-adic string literature (though we will necessarily differ at some important steps). Let us then begin with a "first principles approach" and then show that this does reduce in suitable approximations to a notion of a p-adic open string theory worldsheet. We will then show that the proposed notion is flexible enough to accommodate the structures expected from open string theory, such as Chan-Paton factors.

Compared with the case of the p-adics, Berkovich space is path connected, so we can already see that it makes sense to discuss paths $\gamma : [0,1] \to \Sigma$, where Σ is some p-adic analytic curve, as we would associate with a closed string worldsheet. To get a natural notion of a "boundary" we need to find a way to pick out distinguished paths. One way to proceed, at least for $\mathbb{P}^1_{\mathrm{Berk}}$, is to note that there is a "hyperbolic space" $\mathbb{H}_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(\mathbb{C}_p)$. The condition that we have something "one-dimensional" can then be taken as those points of $\mathbb{C}_p$ invariant under $\mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$, i.e., just the p-adic numbers $\mathbb{Q}_p$. This would get us close to the proposal for the worldsheet of the p-adic open string, but it does come with some obvious drawbacks. For one, this "boundary" is not path connected, and as already remarked, there is no total ordering of the elements, only a partial ordering as dictated by just taking the p-adic norm of elements, as would be in line with our discussion of "radial quantization" in the p-adic setting.

That being said, there is a sense in which we can interpret the standard open string amplitude in terms of a formal contour integral. As explained in Appendix W, the main point is that for the most part, the contributions to the four-point open string amplitude can be reinterpreted as an integral over $\mathbb{Q}_p$, in which we introduce a "radial contour" γ_{rad} consisting of points $x_m = p^m$ for $m \in \mathbb{Z}$, as well as another contour γ_1 associated with the

[84]In the Archimedean setting this of course makes sense because $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$. That is not so for $\mathbb{C}_p$ viewed as an extension of $\mathbb{Q}_p$.

point $1 \in \mathbb{Q}_p$. So, while this prescription is suggestive, it leaves open important conceptual issues such as the ordering of Chan-Paton matrices in more general open string amplitudes.

On the other hand, we have also noted that Berkovich space is path connected, which distinguishes it from both $\mathbb{Q}_p$ and $\mathbb{C}_p$. This in turn means that we can fix a particular closed path $\gamma : [0,1] \to \mathbb{P}^1_{\text{Berk}}$, and use this to define a notion of "worldsheet boundary". The correspondence with actual boundaries obtained from the Berkovich topology, is, however, often imperfect. For example, a ball $\mathbb{B}_R$ of radius R consisting of points $|z| \leq R$ has a boundary which may consist of only a finite number of points. Additionally, an annulus specified by points $r \leq |z| \leq R$ is, in Berkovich space, still simply connected! Again, this is due to the fact that the space is tree-like. All of this means that we must be flexible in how we specify a preferred notion of worldsheet with boundary.

Nevertheless, the existence of a path connected Berkovich space means that once we find an appropriate contour $\gamma : \mathbb{R} \to \mathbb{P}^1_{\text{Berk}}$, ordering of Chan-Paton factors in the associated Koba-Nielsen formula will also follow. To illustrate, since our vertex operators depend on coordinates $x \in \mathbb{P}^1_{\text{Berk}}$, we can speak of the trajectory $x(t)$, and so can label each vertex operator $V_i(t_i)$ as referencing a coordinate t_i for $i = 1, ..., M$ in an M-point amplitude. This is already helpful, because we can now specifically reference a notion of vertex operator ordering on our designated path, and so can also consider multiplication of Chan-Paton matrices in the order they appear along our trajectory. Much as we would with the Archimedean string, we switch the order of multiplication once two points t_i and t'_i cross on the contour.

Summarizing the discussion so far, we have seen that the "standard" p-adic string amplitude makes reference to a formal notion of a contour integral on $\mathbb{Q}_p$, while Berkovich space readily provides us with a notion of connected paths.

Our plan will be to combine these two considerations by making use of an appropriate tropicalization map $\Sigma \to \Sigma_{\text{trop}}$, as induced by the map on affine n-space:

$$\text{Trop: } \mathbb{A}^n \to \mathbb{R}^n \tag{20.40}$$

$$(z_1, ..., z_n) \mapsto (-\log|z_1|, ..., -\log|z_n|). \tag{20.41}$$

As reviewed in Appendix V, this leads to a "skeleton" $\Sigma_{\text{trop}} \to \Sigma$ which embeds in Σ. Our plan will be to interpret this one-dimensional graph as specifying a generalized notion of the worldsheet boundary.[85]

To illustrate, let us consider the affine line $\Sigma = \{x + y = 1\}$ in $\mathbb{A}^2$, and its associated map to $\mathbb{R}^2$. We view the projectivization of Σ as the variety (after a further analytification) associated with the closed string worldsheet at tree level in string perturbation theory. The

[85]Observe that in the Archimedean setting over $\mathbb{C}$, this leads to a map $z \mapsto -\log|z|$ which amounts to mapping $z = re^{i\theta}$ to just its radial component. This produces an ordered set (one point for each radius), but it is not quite the standard worldsheet boundary specified by the real locus $\text{Im}\, z = 0$. Of course, by channel duality we can relate the two.

tropicalization is spanned by the point sets:

$$\Sigma_{\text{trop}}(\mathbb{C}_p) = \left\{(-\log|x|, -\log|1-x|) \subset \mathbb{R}^2 \quad \text{with} \quad x \in \mathbb{C}_p\right\} \tag{20.42}$$

$$\Sigma_{\text{trop}}(\mathbb{Q}_p) = \left\{(-\log|x|, -\log|1-x|) \subset \mathbb{R}^2 \quad \text{with} \quad x \in \mathbb{Q}_p\right\}, \tag{20.43}$$

where here we have referenced the different choices of a ground field. There are a few distinguished points, given by the regions near $x = 0$, $x = 1$ and $x = \infty$, which produce the asymptotic points in $\mathbb{R}^2$ given by $(\infty, 0)$, $(0, \infty)$, and $(-\infty, -\infty)$, respectively (see figure 12). There is also a distinguished midpoint, as given by all points x' of norm one such that $|1 - x'| = 1$, all of which map to the point $(0, 0)$ in $\mathbb{R}^2$. Observe now that the previously mentioned "formal contours" γ_{rad} and γ_1 map to segments of Σ_{trop}. For example, the formal contour γ_{rad} consisting of points $x_m = p^m$ starts at $(\infty, 0)$ and passes through $(0, 0)$, continuing on to $(-\infty, -\infty)$. Meanwhile, the formal contour γ_1 consists of points $1 - p^m$ for $m > 0$ which starts at $(0, \infty)$ and continues down to the origin $(0, 0)$. In this case, there is also a natural contour which passes through all the relevant points. All that is required is that we weight each factor by 1/2. This path proceeds along the contour $(\infty, 0) \to (0, 0) \to (0, \infty) \to (0, 0) \to (-\infty, -\infty) \to (0, 0) \to (\infty, 0)$. Now, although in Σ_{trop} this involves passing through each point more than once, the embedding in Σ can clearly lift to different points, by maps such as $x \mapsto -x$. So in other words, we can lift this path in $\gamma_{\text{trop}} \subset \Sigma_{\text{trop}}$ to a single path $\gamma : [0, 1] \to \Sigma$ where the image of γ under the tropicalization map is just γ_{trop}. This accomplishes the desired path ordering, for an arbitrary number of vertex operator insertions.

A pleasant feature of the present prescription is that it readily generalizes to more complicated worldsheet topologies. Again, the main idea is that we start with a curve Σ and its analytification Σ_{an}. The tropicalization of $\Sigma(\mathbb{Q}_p)$ produces a natural notion of a "worldsheet boundary", as specified by $\Sigma_{\text{trop}}(\mathbb{Q}_p)$. Passing along each leg of the associated skeleton in a path orderered fashion, we get an ordering of Chan-Paton matrices. Note also that by breaking up the open string worldsheet into such finite segments, we can evaluate the results much as we would in the standard Archimedean string.

20.6 Adding D-branes

In the Archimedean string, D-branes play an important role in accessing the non-perturbative structure of string theory (see e.g., [273]). From the perspective of the worldsheet, these arise by imposing Dirichlet boundary conditions along some of the directions of the string worldsheet, e.g, letting $\sigma \in [0, \pi]$ denote the spatial coordinate of the string worldsheet, a D8-brane in a 10D spacetime is specified by the boundary conditions $\phi^9|_{\sigma=\pi} = 0$ and $(\partial - \overline{\partial})\phi^A|_{\sigma=\pi}$ (accompanied by suitable conditions for worldsheet fermions).

Since we have developed a corresponding Berkovich open string, it is natural to ask whether we can extend our considerations to include D-branes. In the context of the p-adic

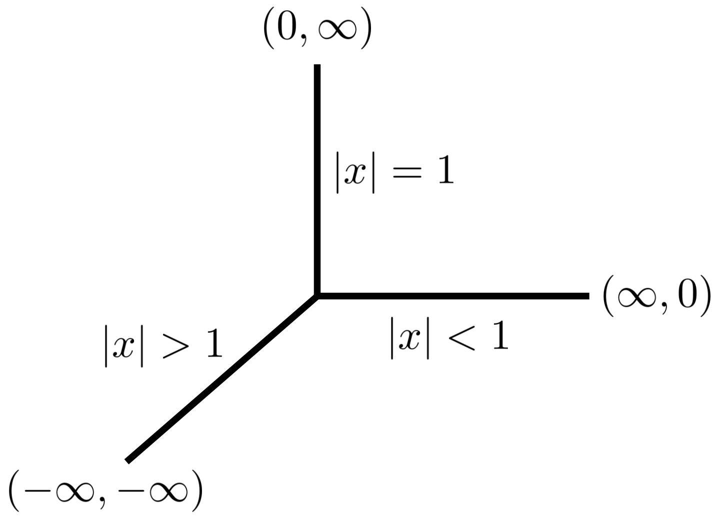

Figure 12: Depiction of the tropicalization of the affine line $\Sigma = \{x+y=1\}$ in $\mathbb{A}^2$. The tropicalization Σ_{trop} is induced by the map $\mathbb{A}^2 \to \mathbb{R}^2$ given by $(x,y) \mapsto (-\log|x|, -\log|y|)$. This leads to a (p,q) web of the sort familiar to physicists in the study of five-brane webs and toric geometry. Here, we use this tropical geometry to define a path connected subspace in Σ_{an} which we can use to specify open string amplitudes with Chan-Paton factors included.

open string, reference [65] interpreted the p-adic open string theory as ending on a spacetime filling D-brane, i.e., the special case of all Neumann boundary conditions.[86] Our aim in this section will be to study the related question in the broader context where we also allow Dirichlet boundary conditions along subspaces in the target space geometry.

To keep things streamlined, we mainly focus on the bosonic sector of the Berkovich string since we expect there to be a natural extension of our discussion to include fermionic degrees freedom. With this in mind, we shall consider Berkovich space worldsheets Σ_{an} with "boundaries". The main idea is already conveyed by returning to the hyperbolic space $\mathbb{H}_{\text{Berk}} = \mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{C}_p)$, which can be viewed as having boundary $\mathbb{P}^1(\mathbb{C}_p)$, and which (as already mentioned previously) has a "one-dimensional" boundary $\mathbb{Q}_p$ given by the fixed point locus of $\text{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$. Since we have already shown (via tropicalization) how to pick out a path ordered contour $\gamma : \mathbb{R} \to \mathbb{P}^1_{\text{Berk}}$, we can break up this path into smaller segments, i.e., we write the interval $[0,1]$ as the union of smaller intervals $I_j = [e_j, e_{j+1}]$ with $e_1 < ... < e_{n+1}$

[86]Even here, however, one can ask whether contributions from closed p-adic strings can be taken into account. The issue is that at least in the Archimedean setting, the endpoints of an open string can "join up" and in so doing produce a close string. This then leads back to earlier questions about how to properly formulate closed p-adic strings. Our philosophy here has been to build a closed string theory via Berkovich space, since this seems to track better with the analytic structure present in the Archimedean setting.

and $e_1 = 0$ and $e_{n+1} = 1$:

$$[0,1] = \bigcup_{j=1}^{n} I_j. \tag{20.44}$$

On each such interval, one can then consider the image set $\gamma(I_j) \subset \mathbb{P}^1_{\text{Berk}}$. We can then speak of a Dr-brane[87] On this segment, we can impose Dirichlet and Neumann boundary conditions via the following conditions:

$$\text{Dirichlet}: \quad \phi^A|_{L_j} = 0 \;\; \text{for} \;\; A = r+1, ..., 10 \tag{20.45}$$

$$\text{Neumann}: \quad (d'^{\phi^A} - d'')|_{L_j} = 0 \;\; \text{for} \;\; A = 0, ..., r. \tag{20.46}$$

The second condition amounts to requiring a match of the different superdifferentials of the closed string worldsheet at the boundary. Note also that we can equip each segment with a suitable set of Chan-Paton factors, so we automatically also get a vector bundle structure on each of our Dr-branes. The point we wish to emphasize here is that the analytic structure present in Berkovich space affords us with a way to make much closer contact with what is encountered in the Archimedean setting.

At this point, it should be clear that we can carry over much of the analysis present in the Archimedean setting, including the study of curved backgrounds, and D-branes of various dimensional support wrapped on suitable cycles. One can then ask, of course, whether the resulting D-brane worldvolume theories have any resemblance to what is observed in the Archimedean setting. It is well-known that even the target space action for p-adic string theories exhibit some non-localities (see e.g., [65]), so we expect this to carry over to the treatment of D-brane effective actions as well. In principle, this can be analyzed by carrying out a study of vertex operator correlators in the Berkovich string, but we leave the study of this for future work.

20.7 Back to Mixed Characteristic

Having seen that we can construct a Berkovich string with target space an Archimedean geometry, it is natural to ask whether we can turn the discussion around once more and also produce string theories where the target space is also a Berkovich space. In particular, one could then establish a p-adic version of objects such as closed strings, open string and D-branes.[88] Performing suitable restrictions to $\mathbb{C}_p$, $\mathbb{Q}_p$ as well as residue fields such as $\mathbb{F}_q$ and $\mathbb{F}_p$, this would also provide a way build more exotic target spaces.

To accomplish this, we begin with two K-analytic Berkovich spaces X and Y, as well

[87]Unfortunately, p has been reserved for other notions so we speak of a Dr-brane as one which fills r spatial directions, as well as the temporal direction, namely it is Dirichlet in the $10 - (r+1)$ directions transverse to the brane. Similar considerations clearly extend to curved backgrounds.

[88]The attempt presented here to formulate this more precisely was prompted by questions from L. Borisov and R. Nally.

as an Archimedean geometry Z.[89] For each of these, we can contemplate maps of the form $\phi_{Z,X} : X \to Z$ as well as $\phi_{Z,Y} : Y \to Z$.[90] We treat X as our worldsheet (i.e., it has dimension one) while Y is permitted to have arbitrary dimension. Clearly, we can also consider morphisms $\phi_{Y,X} : X \to Y$. Observe also that morphisms for the varieties defined over $\mathbb{Q}_p$ naturally extend to this setting. Now, we are interested in constructing a string theory with target space Y rather than just Z. The main issue we face is how to set up a suitable action principle. A natural way to proceed is to fix a particular map $f_{Z,Y} : Y \to Z$, and then restrict to maps which factor through Y, namely we only consider $\phi_{Z,X} : X \to Z$ with $\phi_{Z,X} = f_{Z,Y} \circ \phi_{Y,X}$, where in the path integral we sum over all possible $\phi_{Y,X} : X \to Y$ but do not vary $f_{Z,Y}$. For example, this allows us to define an action principle much as we already did for the Berkovich string:

$$S[\phi_{Y,X}, f_{Z,Y}] = \int\limits_{\mathbb{P}^1_{\text{Berk}}} dx \; G^{(Z)}_{AB} d' \phi^A_{Z,X} d'' \phi^B_{Z,X}, \tag{20.47}$$

in the special case where $X = \mathbb{P}^1_{\text{Berk}}$, and $G^{(Z)}_{AB}$ is the metric on the Archimedean target space Z. Here, we have used the notation $d'_{Z,X}$ and $d''_{Z,X}$ to indicate the superdifferentials associated with maps $X \to Z$ and their corresponding tangent spaces TX and TZ. We will shortly need a generalization of this for higher-dimensional K-analytic spaces Y and their maps $Y \to Z$. We leave this somewhat implicit in what follows since we assume that we can consider a patch of Y which looks like an affine K-analytic space $\mathbb{A}^m_{\text{Berk}}$. Now, since each $\phi^A_{Z,X}$ is specified by composition, we expect that there is a chain rule in play which allows to re-write this in terms of a "pullback metric" on Y. Indicating this metric as:

$$S[\phi_{Y,X}, f_{Z,Y}] = \int\limits_{\mathbb{P}^1_{\text{Berk}}} dx \; g^{(Y)}_{\mu\nu} \partial' \phi^\mu_{Y,X} \partial'' \phi^\nu_{Y,X}, \tag{20.48}$$

where ∂' and ∂'' are superdifferentials defined for maps $X \to Y$, and we have also introduced the pullback metric:

$$g^{(Y)}_{\mu\nu} = G^{(Z)}_{AB} \partial'_\mu f^A_{Z,Y} \partial''_\nu f^B_{Z,Y}. \tag{20.49}$$

The machinery for constructing vertex operators for a Berkovich string with target Y then proceeds as expected. Note also that the construction of D-branes also makes sense in this setting, provided we have the superdifferentials ∂' and ∂'' for maps $X \to Y$, since we can then speak of both Dirichlet and Neumann boundary conditions (much as we did for Archimedean valued maps).

That being said, an unpleasant feature of the above construction is that it requires us

[89]Here, we use the term somewhat loosely. All we mean a space defined over $\mathbb{R}$, possibly either a pseudo-Riemannian or Riemannian manifold. In other words, it is the standard target space for a string theory.

[90]The "backwords ordering" on the subscript is introduced here to make later composition maps more intuitive (at least to us).

to specify a choice of Archimedean target Z, as well as a choice of map $f : Y \to Z$. Presumably our construction should not really depend on these choices, but at the moment the requirement that we can really "extract a number" from our action principle seems to require it. There are, however, at least a few natural choices which immediately present themselves.

One option would be to take $Z = \mathbb{R}^M$ and just fix f as the tropicalization map already encountered. This choice is canonical in the sense that much of our earlier considerations implicitly made reference to tropicalization. Another option would be to take Z to be $\mathbb{C}^\times$, and to view f as a "character map". At least for simple choices of Y such as affine space, or some embedding in affine / projective space, we can induce a character map by restriction from the ambient geometry, so this yields another way to fix both f as well as Z.

Is there anything to favor one such option over another? One demand we can make is that starting from Berkovich space, a suitable "coarsening operation" should apply which allows us to return to the treatment presented earlier in this note. To illustrate, observe that we have canonical embeddings:

$$Y_{\text{Berk}} \leftarrow Y(\mathbb{C}_p) \leftarrow ... \leftarrow Y(L/\mathbb{Q}_p) \leftarrow ... \leftarrow Y(\mathbb{Q}_p) \tag{20.50}$$

$$X_{\text{Berk}} \leftarrow X(\mathbb{C}_p) \leftarrow ... \leftarrow Y(L/\mathbb{Q}_p) \leftarrow ... \leftarrow X(\mathbb{Q}_p) \tag{20.51}$$

In particular, starting from a morphism $X(\mathbb{Q}_p) \to Y(\mathbb{Q}_p)$, we can extend it to a morphism $X_{\text{Berk}} \to Y_{\text{Berk}}$.[91] So, we can opt to restrict our path integral to such restricted sums, and this moves us much closer to the finite sums considered in characteristic p. Indeed, note also that for a morphism $X(\mathbb{Q}_p) \to Y(\mathbb{Q}_p)$ presented locally as a collection of polynomials over $\mathbb{Q}_p$, we can clear denominators and view them as polynomials over $\mathbb{Z}_p$. Passing to the residue field $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$, we return to nearly the beginning. One can also perform a similar operation for $L/\mathbb{Q}_p$ and its residue field. In this extreme limit, however, it is unclear to use whether the tropicalization map $Y(\mathbb{F}_q) \to \mathbb{R}^M$ has any meaning at all (since $\mathbb{F}_q$ has no norm operation). Observe, however, that character maps involving $Y(\mathbb{F}_q) \to \mathbb{C}^\times$ still make sense. At least in this sense, this suggests using a character map operation to construct mixed characteristic / finite characteristic strings and D-branes.

To illustrate how this would work in practice, let us fix Y a subvariety in $\mathbb{A}^m$ and $Z = \mathbb{C}^\times$. Each point in Y can be viewed as a point $(y_1, ..., y_m) \in \mathbb{A}^m$. Now, on the locus $(\mathbb{C}_p)^m$ we can specify a collection of character maps induced from $\chi : \mathbb{C}_p \to \mathbb{C}^\times$, so we can take the product over all the characters to obtain a map:

$$\mathbb{A}^m \to \mathbb{C}^\times \tag{20.52}$$

$$(y_1, ..., y_m) \mapsto \prod_{j=1}^{m} \chi(y_j). \tag{20.53}$$

[91] How unique is such an extension? We do not know.

Viewing $\mathbb{C}^{\times}$ as a flat cylinder, we can take the metric on $\mathbb{C}^{\times}$ to be given by:[92]

$$ds^2_{\mathbb{C}^{\times}} = \frac{dz d\overline{z}}{z\overline{z}}, \tag{20.54}$$

i.e., we have $G_{z\overline{z}} = G_{\overline{z}z} = |z|^{-2}/2$, in the obvious notation.

An interesting feature of this construction is that whereas our previous treatment required the path integral phase factor $\exp(iS/\hbar)$ to also be a character map, the present construction would seem to produce a mod p action with no corresponding phase factor, and would therefore be ill-defined. At the moment we do not see a cleaner way to proceed than just inserting "by hand" an explicit character map operation (i.e., loosely speaking a factor of i) once we reduce to a finite field. Indeed, the statistical field theory interpretation in finite characteristic would suffer from the same ambiguities already pointed out in section 4.6.

[92]One can switch to coordinates $z = \exp(u)$, with $ds^2_{\mathbb{C}^{\times}} = du d\overline{u}$.

21 More General Arithmetic Structures

In sections 16 and 20 we extended our analysis to cover more general numbers given by $N = 2\pi\hbar$ a prime power. In this section our aim will be to generalize these notions to consider the more general case where N has distinct prime factors, writing $N = p_1^{a_1}...p_m^{a_m}$.

To study this case, we continue with our theme of interpreting physical structures in the language of arithmetic geometry. We begin by considering the special case:

$$N = p_1...p_m, \tag{21.1}$$

namely the power of any prime factor is at most one. Thankfully, the relevant structures have also been developed by mathematicians. The main idea we want to exploit is to view the integers $\mathbb{Z}$ as a coordinate ring for a "curve".[93] In algebraic geometry, we construct our geometric spaces by dealing with the spectrum of the ring, namely the set of maximal prime ideals. In the present context, this is just the ideals generated by the primes as well as 0:

$$\operatorname{Spec}\mathbb{Z} = \{\langle p\rangle \text{ for } p \text{ a prime}\} \cup \{0\}. \tag{21.2}$$

In this way of thinking, $\operatorname{Spec}\mathbb{Z}$ is just an affine curve with points specified by these maximal ideals. The meaning of an integer such as $N = p_1^{a_1}...p_k^{a_k}$ is then that it is a collection of k points. Moreover, the exponent a_i indicates the "thickness" of that point as a non-reduced scheme. See Appendix D for a brief discussion on some geometric aspects of $\operatorname{Spec}\mathbb{Z}$.

For a given physical field configuration, then, we can view our action S as a function over the curve $\operatorname{Spec}\mathbb{Z}$, namely, the affine line, where we perform a formal continuation at the origin. We can, of course, consider localization near any prime factor of N, and this will lead us to a power series expansion in that prime. We denote the resulting subscheme of $\operatorname{Spec}\mathbb{Z}$ as $(\operatorname{Spec}\mathbb{Z})_N$.

With this in mind, we considering fibering all of the construction developed previously for a fixed prime p,[94] to construct a larger spacetime and target space:

$$\begin{array}{ccc} X \longrightarrow \widetilde{X} & & Y \longrightarrow \widetilde{Y} \\ \quad\ \ \downarrow & , \quad \text{and} \quad & \quad\ \ \downarrow \\ \operatorname{Spec}\mathbb{Z} & & \operatorname{Spec}\mathbb{Z} \end{array} . \tag{21.3}$$

Each stalk of the fiber is meant to be interpreted as a variety in characteristic p. We can further supplement this by working over different finite fields such as $\mathbb{F}_q$.[95]

[93] Typically, one makes sense of curves over a field. Here, we are relaxing even this condition, so the geometric picture will suffer somewhat. We will nevertheless persist with this language since it is helpful.

[94] We thank R. Donagi for emphasizing this feature of reduction modulo p to us.

[95] As well as further lifts to varieties in mixed characteristic as well as suitable analytifications, we return to this later on.

Now, extending the spacetime and target space in this way, we see that there is a sense in which it is actually more naturally to perform a path integral over all maps from $\widetilde{X}$ to $\widetilde{Y}$. In terms of the geometry of $\operatorname{Spec}\mathbb{Z}$, this means that we instead take a product over all possible prime numbers, but also all maps between these primes by treating $\hbar$ as a map:

$$2\pi\hbar : \operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}, \tag{21.4}$$

where we assume $2\pi\hbar(0) = 0$. This removes the restrictions on exponents introduced in equation (21.1).

A path integral phase factor which includes these effects is given by:

$$\prod_{\hbar}\prod_{p}\prod_{x\in X_p} e^{(iS_x/\hbar(p))} e^{iS_{\text{extra}}[\hbar,x]} \simeq \prod_{x\in X_N} e^{\left(\frac{2\pi i}{N} S_x^{\text{eff}}\right)}, \tag{21.5}$$

where in the above, the additional term S_{extra} also captures more general possible hopping terms between primes. On the righthand side of this expression we view the appearance of a fixed N as the result of performing this product, with an effective action S^{eff} encapsulating these effects. Here, X_N simply refers to the restriction of the fibers of $\widetilde{X}$ to the subscheme $(\operatorname{Spec}\mathbb{Z})_N$. To get a good approximation of S, we can also adopt a Wilsonian perspective, integrating over primes p, starting with the small ones, and then moving to the bigger ones. This provides a sense in which we can pass from short distances back to long distances.

Operator correlation functions also generalize. We view a physical field $\phi(p, x_p)$ as having support over the total space $\widetilde{X}$ given as the fibration of X over $\operatorname{Spec}\mathbb{Z}$. For each prime factor, the definition of the operator is specified in exactly the same way, the only issue is that we should now write a local operator such as the one of line (3.16) as:

$$U(p, x_p) = \exp\left(\frac{i}{\hbar(p)}\phi(x_p)\right), \tag{21.6}$$

with $p \in \operatorname{Spec}\mathbb{Z}$ and $x_p \in X_p$.

Clearly, evaluating correlation functions with this sort of procedure can quickly become unwieldy. A well-motivated approximation is obtained by restricting $2\pi\hbar$ to a special class of maps of the form:

$$2\pi\hbar : \operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z} \tag{21.7}$$

$$x \mapsto x^n. \tag{21.8}$$

The reason these maps are "special" is that they simply send an ordinary point to a "fat point" of the same type, just increasing its multiplicity. This affords us a notion of locality on the primes, so it seems reasonable to make this further restriction.

At a practical level this also makes the reduction over a given prime more tractable, but still quite flexible. In this case, the path integral phase factor collapses to:

$$\prod_{n\in\mathbb{N}}\prod_{p}\prod_{x\in X_p} e^{\left(\frac{2\pi i}{p^n}S_x\right)}. \tag{21.9}$$

Additionally, the structure of operators such as those of line (21.6) also simplifies, and the reduction mod p^n can now be applied fiberwise.

All of the remarks about the emergence of a p-adic and real topology also apply to this enlarged setting. Assuming there is an approximation of the full path integral in terms of an effective action and some integer $N = p_1^{a_1}...p_k^{a_k}$, we can consider each prime factor individually and then the large N limit amounts to assuming that the exponents a_i are all sufficiently large. We thus expect this to produce the expected real topological structure in the continuum limit. Similar comments apply for the corresponding analytifications of the fibers. An interesting feature of working over $\mathbb{C}$ and $\mathbb{C}_p$ is that we also have (non-canonical) isomorphisms $X(\mathbb{C}) \simeq X(\mathbb{C}_p) \simeq X(\mathbb{C}_{p'})$. Of course, the structure of the corresponding analytifications will be different, but this can be viewed as related to how we take a suitable large N limit, as per our discussion in section 20.

21.1 Zeta Functions Revisited

As we have already remarked, evaluating the full path integral of line (21.5) is somewhat unwieldy, but at least provides a general framework for recovering continuum notions of spacetime. Now, in the context of applications to number theory, the idea of starting with a general algebraic variety over $\mathbb{Q}$ and recasting this as a scheme over $\mathbb{Z}$ is a well known procedure. In this context, reduction modulo a prime p then provides important arithmetic information on the behavior of the geometry. For a generic algebraic variety, reduction with respect to a generic prime p will produce a non-singular variety over $\mathbb{F}_p$, and in such cases we can speak of the corresponding Zeta function $Z_{V,p}(z)$, and its relation to a supersymmetric index:

$$\sum_{n\geq 1}\mathrm{Tr}_n\left((-1)^{\mathbf{F}}z^n\right) = \log Z_{V,p}(z) = \sum_{n\geq 1}\#V(\mathbb{F}_{q^n})\frac{z^n}{n}. \tag{21.10}$$

We have also argued that this quantity can also be interpreted as the supersymmetric index of a suitable characteristic p quantum mechanics problem.

Reduction modulo p may also result in a singular space, and this forms the basis for defining the theory of the "conductor" of a variety. For generic varieties this sort of singular reduction happens for a finite number of primes. We have also seen in subsection 13.3 that on physical grounds, the structure of the supersymmetric index should allow us to make sense of the Zeta function, even if there is a singular reduction.

Now, in the more general setting just introduced, we have been considering a further fibration over $\operatorname{Spec}\mathbb{Z}$, so we can speak of the supersymmetric index obtained from working with all the different primes. If we restrict to the special case where the map $2\pi\hbar$ of line (21.7) is just the identity map (no fat points at all) then we just take the product over all the different zeta functions. This produces the expression:

$$Z_{V,\mathbb{Q}}(\{z_p\}) \equiv \prod_p Z_{V,p}(z_p). \tag{21.11}$$

In the arithmetic geometry literature it is customary to work in terms of a single uniform fugacity. Introducing a complex number s (with suitable domain of definition to ensure convergence of the product), we can write $z_p = p^{-s}$. Note that this means larger primes are "penalized" in the associated partition function. Making this substitution, we arrive at the Zeta function:

$$\zeta_{V,\mathbb{Q}}(s) \equiv \prod_p \zeta_{V,p}(s), \tag{21.12}$$

where we changed notation ($\zeta = Z$) to emphasize the different variable dependence.

The present formulation also suggests some natural generalizations of these sorts of formulae. Instead of just dealing with the identity map for $2\pi\hbar : \operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}$, we can consider more general powers. Evaluating the corresponding local Zeta functions and taking the product over prime powers provides a more general set of objects to study.

21.2 Geometric Engineering Revisited

It is also interesting to revisit our discussion of geometric engineering, especially as a way to formulate a suitable notion of gauge theory on the bigger space $\widetilde{X} \to \operatorname{Spec}\mathbb{Z}$, and thus as a way to formulate mathematical (and physical!) quantities of interest.

The general geometric notions we need are specified by working with arithmetic schemes. The idea is to fix some variety V over an algebraic number field K (i.e., some finite field extension of the rational numbers $\mathbb{Q}$), and consider the ring of integers $\mathcal{O}_K$. Then, we can consider the fibration $V \to \operatorname{Spec}(\mathcal{O}_K)$ as obtained by reduction of the variety at prime ideals of $\mathcal{O}_K$. Doing so, the total space has "one dimension more" than the reduction. Clearly, this is precisely the situation outlined previously in the special case where $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$. A simplified but still important example is provided by the case of algebraic curves over K. In this case, the total space is referred to as an arithmetic surface S. See figure 13 for a depiction.

Our aim in this brief subsection will be to sketch how to use notions of geometric engineering to sketch a formulation of a gauge theory on an arithmetic surface S. We expect that similar notions also work for more general arithmetic varieties since geometric engineering also extends to this broader setting. We will also find it convenient to allow K to sometimes differ from $\mathbb{Q}$, even though we expect the physically most interesting case is likely provided by the "simplest situation." One reason for doing this is that in previous discussion of geometric engineering in characteristic p, we saw that the sharpest analogy with the characteristic zero case suggests working with the algebraic closure $\overline{\mathbb{F}}_p$.

To formulate gauge theory on S, we view it as an arithmetic surface of ADE singularities X such that the total space is Calabi-Yau. We note that such a notion makes sense because even in the setting of an arithmetic variety defined over a ring of integers such as $\mathcal{O}_K$, we can still speak of a canonical sheaf (see reference [274]). To be explicit, consider the special case of an A-type singularity. Then, we would write:

$$y^2 = x^2 + u^N, \tag{21.13}$$

where the locus $x = y = z = 0$ defines S. Of course, in the "standard setting," we would view this as engineering the Vafa-Witten system [188] as familiar from model building in F-theory (see e.g., [275, 171, 276]). The non-trivial step in the characteristic zero setting has to do with ensuring that the two notions of moduli spaces from gauge theory and singular Calabi-Yau geometry actual specify the same degrees of freedom.

Here, we will simply use the geometry as a way to *define* what we could possibly mean by gauge theory on the arithmetic surface, leaving a complete treatment for future work. In accord with the usual characteristic zero case, we expect to have a notion of a vector bundle $\mathcal{E}$ with an A-type structure group $SL(N, K)$, and a Higgs field. Now, to see the appearance of the vector bundle in our setting, we observe that we can perform blowups of X, even in the arithmetic setting. Doing so, we observe the appearance of "fibral divisors" $D_1, ..., D_N$. These can be viewed as divisors of an ADE singularity, which is then further fibered over S.[96] We can now see the appearance of the 2-cycle class $[S]$ in the intersection pairing:

$$D_i \cdot D_j = C_{ij}[S], \tag{21.14}$$

where C_{ij} with $i, j = 1, ..., N$ is the usual intersection theoretic Cartan matrix (-2's on the diagonal). Taking linear combinations of effective divisors, we then build up the usual notion of a root system fibered over S. Implicitly, then, the deformations of line (21.13) are specifying the Casimir invariants of a Higgs field:

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}_S, \tag{21.15}$$

where the notion of $\mathcal{K}_S$ as the canonical sheaf makes sense for an arithmetic surface, and we view $\mathcal{E}$ as a sheaf on S which admits a group action by $SL(N, K)$. Now, for each prime $p \in \mathcal{O}_K$, we can fix our attention on the corresponding stalk S_p. This is akin to the characteristic p Hitchin system we already encountered.

An interesting feature of this setup is that we can now discuss surface operators, as associated with specifying a prescribed singularity structure for our Higgs field along a curve

[96] Here, the proper notion of divisor, and intersection of divisors implicitly makes reference to Arakelov's intersection theory [277, 278] (see reference [274] for an introduction). This is necessary to stipulate because we need to be able to specify what happens "at infinity," in $\mathrm{Spec}\, \mathcal{O}_K$.

(or curves) in S. In the context of geometric engineering, this has been analyzed for example in references [171–173]. Call one such curve Σ. Then, this also specifies a divisor in S, and the Higgs field has a residue along Σ which is some element in the Lie algebra $\mathfrak{sl}(N, K)$:

$$\mathrm{Res}_C \Phi \in \mathfrak{sl}(N, K). \tag{21.16}$$

In the geometric engineering setting, we can generate such profiles by colliding different singularities together. For example, still within the context of our setup as in line (21.13), introduce another arithmetic surface S' locally specified as the hypersurface $x = y = v = 0$. Then, we can consider:

$$y^2 = x^2 + u^N v^M, \tag{21.17}$$

as associated with $SL(N, K)$ gauge theory on $u = 0$ and $SL(M, K)$ gauge theory on $v = 0$. The surfaces S and S' intersect along $u = v = 0$ which we view as a horizontal Arakelov divisor in S.

In the characteristic zero setting we would say that there are "matter fields on Σ" as specified by elements of $H^0(\Sigma, \mathcal{K}_\Sigma^{1/2} \otimes (\mathcal{E} \otimes \mathcal{E}') \,|_C)$. Presumably there is a suitable concept of this for divisors in arithmetic surfaces, via the analog of theta functions. In the special case where the bulk vector bundles are trivial, there exists a Lie algebra valued pairing for the zero modes (see [171] for the precise definitions) which we can use to specify the value of the Higgs field residue:

$$\mathrm{Res}_C \Phi = \langle \psi^c, \psi \rangle \, . \tag{21.18}$$

So, using concepts from geometric engineering, notions of surface operators (at least for Higgs fields) and gauge theory over S still appear to make sense.

It would also be natural to investigate the effects of S-duality on such gauge theories. Taking our cue from the characteristic zero setting, we can consider a product of an elliptic curve $\mathbb{E}$ and an ADE singularity. For example the elliptic curve $\mathbb{E}$ can be specified in Weierstrass form as $y^2 = x^3 + fx + g$, with $f, g \in K$, and upon choosing an embedding in $\mathbb{C}$, we can also associate it with a fixed choice of complex structure parameter τ. Fibering this geometry over our "4D spacetime" the arithmetic surface S, we can thus assign our gauge theory the standard parameter τ which transforms under $SL(2, \mathbb{Z})$ in the usual way:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \text{ with } ad - bc = 1 \;\; a, b, c, d \in \mathbb{Z}. \tag{21.19}$$

So, at least in principle, the effects of S-duality can be studied in this framework.

21.2.1 Arithmetic Line Operators

Having come this far, we can also ask whether we can set up the usual notions of electric and magnetic line operators in our gauge theory system. To begin, we exploit the structure of S as a fibration over $\operatorname{Spec} \mathcal{O}_K$ to define a formal one-form:

$$A = A_{\mathfrak{p}} d\mathfrak{p} + A_{x_{\mathfrak{p}}} dx_{\mathfrak{p}}, \tag{21.20}$$

where each $\mathfrak{p}$ refers to a maximal prime ideal of $\mathcal{O}_K$ as specified by the sheaf of differentials in $\Omega^1(\operatorname{Spec} \mathcal{O}_K)$ with local differential $d\mathfrak{p}$ and the fiber at each stalk comes with a differential $dx_{\mathfrak{p}}$ as defined on the curve $S_{\mathfrak{p}}$, which is just a curve over the finite field $\mathbb{F}_q \simeq \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$ with $q = p^r$ for some prime p and $r > 0$. We can view A as a one-form valued in $\mathfrak{sl}(N, K)$, but in which we also need to reduce each component in the stalk $A_{x_{\mathfrak{p}}}$ mod the given prime $\mathfrak{p}$.

Our first aim will be to define a notion of an electric line operator. First of all, for the vertical divisors, we are speaking of $A_{x_{\mathfrak{p}}} dx_{\mathfrak{p}}$, and so we can specify holonomies by appealing to the spectral cover construction for a curve over $\mathbb{F}_q$, namely we first take a curve Σ over $\mathbb{F}_q$, and on the spectral cover $\widetilde{\Sigma} \to \Sigma$ we take a line bundle. Under the suitable pushforward map, we then get a vector bundle, which in turn implicitly specifies a line operator for us.

So, the real issue here is to see if we can define a suitable notion of a line operator for a horizontal divisor, as specified by a section $P : \operatorname{Spec} \mathcal{O}_K \to S$. To do this, we first pick an embedding $K \hookrightarrow \mathbb{C}$, and order the primes $\mathfrak{p}$ according to their absolute values, viewed as standard complex numbers. Each prime can then be viewed as living on the geometric cylinder $\mathbb{C}^{\times}$. We can then specify a partial ordering on $\operatorname{Spec} \mathcal{O}_K$, and for ease of exposition we will assume this is a total ordering (as occurs, for example, in the special case $\operatorname{Spec} \mathbb{Z}$). This means we also have a notion of a path ordered exponential. The last ingredient we need to specify is a choice of representation R for $\mathfrak{sl}(N, K)$, and we denote the resulting matrix valued connection as $A^{(R)} = A^a T^a_{(R)}$, where the T^a are the generators of the lie algebra in the representation R. We view the restriction $A^{(R)}_{\mathfrak{p}}$ on each stalk as implicitly specified by a representation on $\mathfrak{sl}(N, \mathbb{F}_q) \simeq \mathfrak{sl}(N, \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)$. We can then define an ordered product:

$$W_R = \prod_{\mathfrak{p}} \exp\left(\frac{2\pi i}{p} A^{(R)}_{\mathfrak{p}}\right). \tag{21.21}$$

Observe that we have inserted pre-factors of $2\pi i/p$. The prime p is the characteristic of the finite residue field $\mathbb{F}_q \simeq \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$. We have included this factor because we need to make sure that W_R makes sense as an operator acting on the Hilbert space defined by each stalk $S_{\mathfrak{p}}$ of the arithmetic surface. In physical terms, the operator W_R is a path ordered exponential, and specifies an electric line operator.

What about the magnetic line operators? At least classically, we see what to do using our geometrically engineered setup. Indeed, since we have an elliptic curve over K with a choice of embedding in $\mathbb{C}$, we can also specify the A-cycles and S-dual B-cycles on the elliptic curve.

In the quantum setting the situation is more subtle because we view the electric line operators as order operators and the magnetic line operators as disorder operators, namely, we prescribe boundary conditions [189]. But we know precisely how to implement this prime ideal by prime ideal in $\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$. Indeed, what we do is examine the intersection of the image of $P : \operatorname{Spec} \mathcal{O}_K \to S$ with each vertical stalk. In the associated curve $S_{\mathfrak{p}}$, we are marking a boundary condition, as specified by a choice of representation in $\mathfrak{sl}(N, \mathbb{F}_q) \simeq \mathfrak{sl}(N, \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)$.

There is a rather rich story explained in [189] for how these operator can act on a 2D space obtained by "dimensional reduction", and this provides a physical basis for the geometric Langlands program. There seem to be some parallels with the story we are setting up here which would be interesting to explore further.

We leave a more complete treatment of these possibilities for future work.

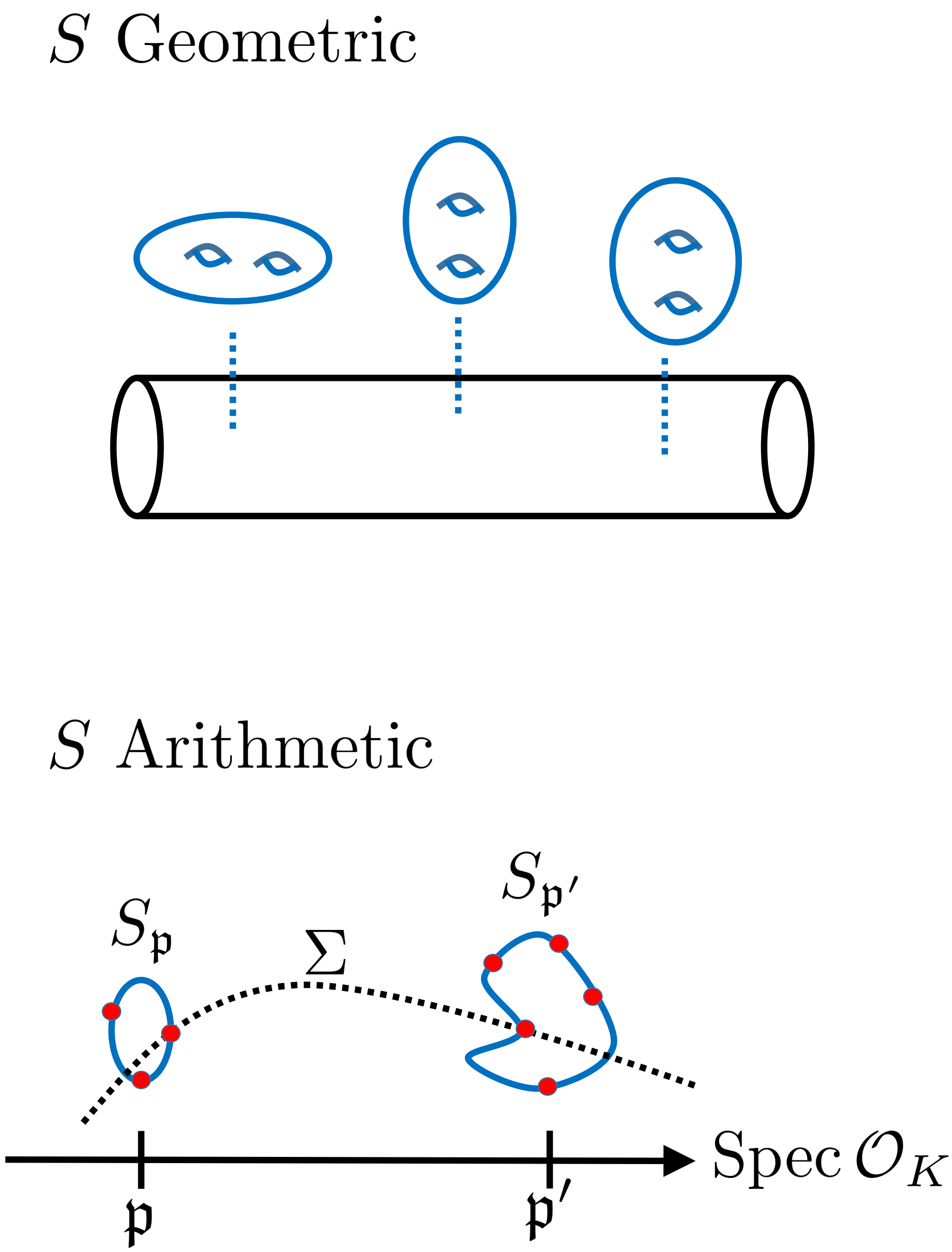

Figure 13: Depiction of a geometric surface (top) in characteristic zero, viewed as a Riemann surface fibered over a cylinder, as well as the analog for an arithmetic surface (bottom) viewed as a fibration over $\operatorname{Spec}\mathcal{O}_K$, with K an algebraic number field and $\mathcal{O}_K$ its ring of integers. For each prime $\mathfrak{p}$, the fiber is given by the reduction of a variety over K over that prime. We have also drawn a depiction of a horizontal divisor Σ, as well as its intersection with some examples of vertical divisors such as $S_{\mathfrak{p}}$ and $S_{\mathfrak{p}'}$, with $\mathfrak{p}$ and $\mathfrak{p}'$ primes of $\operatorname{Spec}\mathcal{O}_K$.

Part IV

The End

22 Conclusions and Further Speculations

In this note we have studied a class of physical systems in which the degrees of freedom are discretized. At some broad level, this can be phrased as taking a different choice of "natural units" in which the reduced Planck constant is instead set to the value $\hbar = N/2\pi$ with N an integer, which we view as a highly quantum regime of a physical system. From this starting point, we have shown that when $N = p$ a prime number, that the resulting physical system can be understood in terms of arithmetic geometry in characteristic p. Additionally, we have seen that this same structure persists when N is a more general integer. We have developed the analog of bosonic and fermionic degrees of freedom, and have also sketched how more general field theories can be written in characteristic p. This allowed us to present a (speculative) physical interpretation of the Hasse-Weil Zeta function. An additional feature of our considerations is that some well-established algebro-geometric correspondences appear to have close characteristic p analogs. This in turn suggests that the highly quantum regime of a string compactification may simply involve relaxing the choice of algebraic field. We have also seen that in suitable limits, more refined topological structures appear to emerge. For example, a suitable analytification procedure allows us to adapt much of the machinery of the Archimedean string to the p-adic analytic setting. In the remainder of this section, we discuss some further speculations, as well as possible areas for future investigation.

We have taken some first steps in understanding the structure of loop corrections. As one might have expected, the discretized nature of our computations leads to well-regulated expressions in which a number of loop corrections identically vanish. We have also indicated that more general loop corrections are indeed possible, again indicating evidence for a non-trivial quantum theory. Along these lines, it would be quite interesting to investigate the structure of renormalization, and in particular the content of renormalization group flow across discretized parameters. One way to set up such an analysis would be to explicitly start integrating out some of the coefficients appearing in our mode expansions, and tracking the resulting effect on correlation functions of the theory.

One of the elements we have hinted at but have not fully developed is the structure of cohomological theories as specified by supersymmetric field theories in characteristic p. It would seem worthwhile to develop this further.

We have also presented a general expectation that some of the characteristic zero correspondences between gauge theories defined on algebraic varieties and singular local Calabi-Yau spaces extend to characteristic p. This is particularly intriguing in light of the physical formulation of the geometric Langlands program, which relies heavily on a topological twist of $\mathcal{N} = 4$ Super Yang-Mills theory [189]. From the perspective of geometric engineering, it is natural to ask whether there is a characteristic p analog of this gauge theory which could be geometrically engineered. Very speculatively, one might use this to provide a physical underpinning for some aspects of the Langlands program. This would be in line with our other remarks that we can formulate a notion of a gauge theory over arithmetic surfaces with reduction along each fiber producing a corresponding characteristic p Hitchin system.

Indeed, reference [72] noted that at least for suitable flux compactifications and their relation to arithmetic Calabi-Yau threefolds, there is a notion of modularity which might persist based on the associated Zeta functions (see also [73]). In the present note we have provided some additional physical motivation for such structures. It would seem interesting to develop this further.

Perhaps more directly, we also sketched how geometric engineering can be used to provide an operational definition of certain gauge theories on an arithmetic surface. Indeed, suitable reductions over a prime in this setting return us to the case of a Hitchin system in characteristic p, so it would seem natural to study the structure of physical notions such as S-duality and its action on electric and magnetic surface operators (perhaps along the lines of [189]).

Continuing in the vein of possible mathematical applications, we have seen that at least when N is a prime number p, that some physical structures can be formulated in terms of the geometry of schemes defined over the finite field $\mathbb{F}_p$ or some extension thereof. Writing $p = 2\pi\hbar$, it is natural to ask whether the physical limit $\hbar \to 1$ has any bearing on questions in arithmetic geometry. This sort of limiting procedure is sometimes mentioned in the context of what could possibly be meant by the finite field $\mathbb{F}_{\mathrm{un}}$, namely, the field with "one element" (see e.g., [279] for a recent discussion). It would be interesting to see whether physical considerations provide a new perspective on these questions.

In terms of "practical applications," it has been well appreciated for some time that elliptic curves over finite fields can be used to build examples of public-key encryption schemes (see e.g., [280–282]). One of the (at least original) motivations for the present work was to see whether the families of elliptic curves used in F-theory compactifications can be transported to the realm of finite fields, thus providing a specific class of encryption schemes involving higher-dimensional varieties defined over finite fields. It would be interesting to revisit some of the classic constructions of F-theory backgrounds with such an application in mind.

The discretization of a physical theory immediately raises additional questions in the context of quantum gravity. For example, in reference [14] it was argued that Newton's constant might be quantized in units of $1/f_\pi^2$, with f_π a mass scale of a non-linear sigma model and in references [15–17], it was argued that the Fayet-Iliopoulos parameter of a supergravity theory might be quantized in units of $2M_{\mathrm{pl}}^2$. An additional hint at the quantization of fundamental parameters appeared in [283], which argued that in appropriate decoupling limits, quantities such as $\alpha_{\mathrm{GUT}}^{-1}$ of a Grand Unified Theory (GUT) might also be discretized. The present note has taken some steps at understanding some examples of this sort, including an analysis of Planckian scale FI parameters. It would be interesting to see whether the discretization of other physical parameters can also be understood using methods from arithmetic geometry.

At a practical level, the construction we have presented has the merit of dealing with systems with discretized degrees of freedom. This in turn means that numerical computations should be possible as well. Since, however, we have mainly had to work with complex phases (at least in the characteristic $p > 0$ case), there is no guarantee that the resulting sums will be numerically stable. On the other hand, we also saw how to setup a statistical field theory formulation, especially in the p-adic and p-adic analytic case. It would be interesting to examine the numerical stability of such computations, especially in comparison with lattice field theories defined in Euclidean space.

A related comment is that while we have emphasized that our field theories typically involve discretized spacetimes and target spaces, the space of morphisms is still infinite. It would be very interesting to consider in detail the matrix model approximation obtained by just working with point set maps. We expect that in this case, there is some loss of fidelity which is only truly recovered in the large matrix limit.

Some of our physical picture can be interpreted in terms of quantum error correcting codes. As a potential practical application of our considerations, it is natural to ask whether our path integral formalism implicitly performs a sweep over candidate quantum error correcting codes. It would be interesting to see whether the resulting quantum error correcting codes are "optimized" in any practical sense.

Our construction of field theories also highlighted that there is a characteristic p analog of the graviton, which we associate with a family of symmetric bilinear forms. We also saw that in characteristic p, there is little meaning to the "signature" of a metric since there is no ordering of elements in $\mathbb{F}_p$. Interpreting the characteristic p limit of a string compactification as the highly quantum regime of gravity, this suggests that in this discretized setting, distinctions between Lorentzian, Euclidean or more general spacetime signatures evaporate.

An intriguing feature of the present considerations is that one can view our construction at its most primitive level as specified by a Grothendieck topology for a suitable category where this a notion of "quasi-locality" even in characteristic p. From this, we saw how the structures of classical and quantum error correcting codes emerge from a suitable adaptation of a path integral defined in characteristic p. We have also sketched how standard continuum physics could emerge from these discretized considerations in a suitable large N limit, both in terms of its connection to p-adic formulations of physics, as well as standard formulations over the real numbers.

Along these lines, we have introduced a new notion of p-adic strings in which worldsheets are associated with p-adic analytic spaces. This appears to retain far more of the structure present in the Archimedean case, and suggests a general template for analyzing the onset of non-local structures in string theory. It is also tempting to generalize beyond the "Berkovich string" to an object such as the "Perfectoid string" (in the sense of Scholze [284], see [285] for an early survey). We do not know how to do this, but it is a natural thing to try next.

One element which the analytification procedure helps to clarify is the sense in which other celebrated structures from the Archimedean setting such as closed strings, open string

Chan-Paton factors, as well as D-branes can be taken into account in the non-Archimedean setting. Moreover, we sketched how, through a sequence of ensuing "coarsening" operations, we can then return all the way to characteristic $p > 0$ geometries. We expect that filling out this procedure more completely would help to bolster / refine some of the other speculations presented elsewhere throughout this note. For example, in the context of geometric engineering of quantum field theories via string compactification, the localized degrees of freedom of the quantum field theory arise from D-branes wrapped on collapsed cycles. With an improved characterization of D-branes in such p-adic analytic spaces, one could more directly track the resulting degrees of freedom.

As we briefly discussed, a curious feature of many topological quantum field theories is the ubiquitous appearance of phase factors in $\mathbb{Q}/\mathbb{Z}$ (namely, complex phases given by a root of unity). Moreover, in many bordism computations, one also finds that the computation is often p-local over a finite number of primes. We sketched how to build quasi-topological actions in the characteristic $p > 0$ setting, so it is natural to expect that in the p-adic analytic setting we can develop the requisite topological field theories as well, including the corresponding cobordism theories.

Concerning S-matrix observables, a related comment here is that recently it was conjectured that much of the mathematical structure of flat space observables such as the S-matrix of quantum field theory can be understood in the general framework of tame functions [286] (for applications in the context of physics, see also [287–290] and for a review of the relevant mathematical structures, see reference [291]). The general line of argument establishing this is to first show that scattering amplitudes are in fact (at least to reasonable loop order) characterized by period integrals of a Calabi-Yau space [292–294, 286, 295]. Since these period integrals are in turn governed by a Pichard-Fuchs differential equation, it is clear, à la Dwork / Candelas and de la Ossa and its generalizations that the same structure carries over to the p-adic setting as well. In particular, it is tempting to reinterpret the appearance of tame geometry as the requirement that relevant observables of quantum gravity remain well-behaved under a change of ground field. This suggests another avenue of investigation which would be exciting to explore.

Indeed, the present algebro-geometric perspective suggests a somewhat different starting point for understanding the physical origins of p-adic strings and p-adic holography, and the potential implications for physics defined over the reals. It would be interesting to develop this further.

But that is enough speculation for one note.

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24 Notes and Version History

Note: These notes were first publicly circulated in May 2020, and were also presented at the Arithmetic Geometry and Quantum Field Theory seminar series in October 2020; and at the Mathematics - String Theory seminar series at the Kavli IPMU in May 2021. A version which is intermittently updated is available at `www.jjheckman.com/research`. If these notes are of interest / figure in your own research, please cite them as appropriate. We would appreciate being alerted to notable omissions in referencing and / or elementary mistakes.

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Part V

Appendices

A 1D Lattice Systems

In this Appendix we present a brief analysis of some discretized 1D systems modulo N. We anticipate that similar formal manipulations are available in more general field theories, which would be amenable to a numerical analysis. Although we have emphasized that the proper framework for doing our computations is based on integrating over the moduli space of morphisms between schemes in characteristic p, for "practical purposes" calculating with respect to a fixed lattice field theory formulation should provide an adequate approximation for many purposes. The more general formulation seems necessary to fully capture the arithmetic geometry associated with these systems. Before proceeding, we should of course remind the reader that what is frequently done in lattice approximations to field theory is to consider discretizing spacetime, but to allow continuum valued fields. Here, we are asking a slightly different question since we are discretizing both the source and the target spaces right from the start.

With all of this in mind, we now consider a lattice formulation for a 1D system with a discretized time direction in which the reduced Planck constant satisfies:

$$\hbar = \frac{N}{2\pi}. \tag{A.1}$$

As we have already mentioned in section 3, this sort of discretization also impacts the time evolution operator, restricting us to discretized spacetimes. Our plan in this section will be to analyze a few explicit lattice systems where we take the time direction $t \in \mathbb{Z}/N\mathbb{Z}$.

Throughout our analysis we shall implicitly assume 2 does not divide N so that 2 is an element of the group $(\mathbb{Z}/N\mathbb{Z})^{\times}$, the multiplicative group of integers modulo N. This is mainly for ease of exposition; one can relax this assumption at the expense of not writing expressions such as $1/2$ in the definition of the action.[97]

So, we take as our action:

$$S[\phi] = \sum_{t \in \mathbb{Z}/N\mathbb{Z}} L[\phi(t)] \tag{A.2}$$

with:

$$L = T - V, \tag{A.3}$$

where the kinetic term is given by:

$$T = \sum_{1 \le i,j \le N} \frac{1}{2} \Gamma_{ij} \phi(i) \phi(j). \tag{A.4}$$

and for now, we do not specify the potential $V(\phi)$. To make things concrete, we shall assume

[97] We do this to maintain contact with the "standard" presentation of Lagrangians, but we caution that its meaning in the arithmetic setting is somewhat different; the expression $1/2$ is represented by the integer $(N+1)/2$.

that Γ is an $N \times N$ matrix which takes the form of a specific 1D lattice Laplacian with boundary conditions at the ends of the lattice:

$$\Gamma = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & ... & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \tag{A.5}$$

Where to illustrate the main ideas we have chosen Γ so that that it does not have zero modes. This case is helpful because we can bypass various subtleties having to do with zero modes. That being said, we can of course make more general choices.

Our plan will be to analyze some aspects of this system. We mainly focus on the case of $N = p$ a prime number, but also discuss a generalization to $N = p^a$, which would correspond to a "fat point" of $\operatorname{Spec} \mathbb{Z}$.

We first consider the case of a $D = 1$ massless free scalar, and then turn to some cases with a potential switched on. Most of the manipulations we use are covered in standard quantum mechanics and quantum field theory textbooks (see e.g., [296–298,297,299,300,141,301]). We include these computations here for the reader unfamiliar with these sorts of manipulations. The main subtlety we encounter will have to do with obtaining a propagator, and analyzing the resulting correlation functions.

A.1 The $D = 1$ Free Scalar

We start with a 1D free scalar reduced modulo $N = p^a$, namely we set $V = 0$. To evaluate correlation functions, we introduce a source term $J(t)$ and study the generating function for correlators:

$$Z[J] = \sum_{\phi(1) \in \mathbb{Z}/N\mathbb{Z}} ... \sum_{\phi(N) \in \mathbb{Z}/N\mathbb{Z}} \exp\left(\frac{2\pi i}{N}\left(\sum_{1 \leq i,j \leq N} \frac{1}{2}\Gamma_{ij}\phi(i)\phi(j) + \sum_{t=1}^{N} J(t)\phi(t)\right)\right). \tag{A.6}$$

A general comment here, already remarked upon in the main body of the text, is that the path integral here clearly truncates to a finite sum, as befits a lattice approximation. In the main body, we have emphasized the important role of having an infinite number of morphisms, but this of course complicates the evaluation and regulation of the accompanying infinite sums. We view expressions such as equation (A.6) as an approximation to the other actions / path integrals studied in the main body.

Now, in characteristic zero, it is natural to expand in "Fourier modes." This presents some complications, especially when reducing mod p^a. Rather than follow this route, we will instead stick to position space.

The first point we want to make is that the determinant of Γ is:

$$\det \Gamma = N + 1 \equiv 1 \bmod N. \tag{A.7}$$

So the inverse matrix with entries in $\mathbb{Z}/N\mathbb{Z}$ makes sense.

We now can write the action with a source term added as:

$$S[\phi] = \frac{1}{2}\Gamma_{ij}(\phi(i) + \Gamma^{-1}_{ii'}J(i'))(\phi(j) + \Gamma^{-1}_{jj'}J(j')) - \frac{1}{2}\Gamma^{-1}_{ij}J(i)J(j), \tag{A.8}$$

where in the above, we have summed over repeated indices. In the above expression we have introduced the inverse matrix Γ^{-1}, which is being computed in $\mathbb{Z}/N\mathbb{Z}$. In particular, we are viewing the entries of Γ^{-1}_{ij} as being in $\mathbb{Z}/N\mathbb{Z}$ rather than $\frac{1}{N+1}\mathbb{Z} \subset \mathbb{Q}$. This is the "natural" choice to make because all of our other quantities, including $\phi(i)$ and $J(i)$ are valued in $\mathbb{Z}/N\mathbb{Z}$.

The integrand of the generating function $Z[J]$ now takes the form:

$$\exp\left(\frac{2\pi i}{N}\left(\frac{1}{2}\Gamma_{ij}(\phi(i) + \Gamma^{-1}_{ii'}J(i'))(\phi(j) + \Gamma^{-1}_{jj'}J(j')) - \frac{1}{2}\Gamma^{-1}_{ij}J(i)J(j)\right)\right). \tag{A.9}$$

The point is that for each $\phi(i)$, we sum over all entries anyway, so the shift by $\Gamma^{-1}_{ii'}J(i')$ is "harmless". So, we can write the generating function as:

$$Z[J] = Z[0]\exp\left(\frac{2\pi i}{N}\left(-\frac{1}{2}\Gamma^{-1}_{ij}J(i)J(j)\right)\right), \tag{A.10}$$

just as we would in characteristic zero. We caution, however, that this similarity is somewhat deceptive since, for instance, the inverse of Γ is computed in $\mathbb{Z}/N\mathbb{Z}$ rather than $\mathbb{Q}$.

Evaluating correlation functions superficially proceeds as in characteristic zero by taking functional derivatives of the sources:

$$\langle \phi(t_1)...\phi(t_m)\rangle = \left(\frac{1}{Z[0]}\frac{\hbar}{i}\frac{\delta}{\delta J(t_1)}...\frac{\hbar}{i}\frac{\delta}{\delta J(t_m)}Z[J]\right)_{J=0}. \tag{A.11}$$

As an example, we have, for $1 \leq s, t \leq N$:

$$\langle \phi(s)\phi(t)\rangle = -\frac{\hbar}{i}\Gamma^{-1}_{st} = -\frac{N}{2\pi i}\Gamma^{-1}_{st}. \tag{A.12}$$

But such expressions are, by themselves, ambiguous because operators such as $\phi(t)$ can be viewed as taking values in the integers, rather than $\mathbb{Z}/N\mathbb{Z}$. We can, however, replace these expressions by correlation functions such as:

$$\langle \exp(2\pi i\alpha\phi(s)/N)\exp(2\pi i\beta\phi(t)/N)\rangle = \exp\left(-\frac{2\pi i}{N}\alpha\beta\Gamma^{-1}_{st}\right). \tag{A.13}$$

Similar considerations hold for higher point correlation functions, via a simple application of Wick's theorem.

It is also interesting to directly analyze the behavior of the propagator Γ^{-1} for different choices of N. To keep things manageable, we compute the inverse for N a prime number. To illustrate, here are the first few inverses:

$$p = 3 : \Gamma^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \tag{A.14}$$

$$p = 5 : \Gamma^{-1} = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 4 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 1 & 3 \\ 2 & 4 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & 2 & 1 \\ -1 & -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 1 & -2 \\ 2 & -1 & 1 & -2 & -1 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} \tag{A.15}$$

$$p = 7 : \Gamma^{-1} = \begin{bmatrix} 0 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 3 & 1 & 6 & 4 & 2 \\ 5 & 3 & 1 & 5 & 2 & 6 & 3 \\ 4 & 1 & 5 & 2 & 5 & 1 & 4 \\ 3 & 6 & 2 & 5 & 1 & 3 & 5 \\ 2 & 4 & 6 & 1 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & -3 & 3 & 2 & 1 \\ -1 & -2 & 3 & 1 & -1 & -3 & 2 \\ -2 & 3 & 1 & -2 & 2 & -1 & 3 \\ -3 & 1 & -2 & 2 & -2 & 1 & -3 \\ 3 & -1 & 2 & -2 & 1 & 3 & -2 \\ 2 & -3 & -1 & 1 & 3 & -2 & -1 \\ 1 & 2 & 3 & -3 & -2 & -1 & 0 \end{bmatrix}. \tag{A.16}$$

As an example where N is not a prime number, here is the propagator over $N = 3^2$:

$$N = 3^2 : \Gamma^{-1} = \begin{bmatrix} 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & 7 & 5 & 3 & 1 & 8 & 6 & 4 & 2 \\ 7 & 5 & 3 & 0 & 6 & 3 & 0 & 6 & 3 \\ 6 & 3 & 0 & 6 & 2 & 7 & 3 & 8 & 4 \\ 5 & 1 & 6 & 2 & 7 & 2 & 6 & 1 & 5 \\ 4 & 8 & 3 & 7 & 2 & 6 & 0 & 3 & 6 \\ 3 & 6 & 0 & 3 & 6 & 0 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \end{bmatrix}. \tag{A.17}$$

A.2 Adding a Mass Term

A common deformation of the free scalar involves adding a quadratic term in the physical fields. Let us consider adding a perturbation such as:

$$V(\phi) = \frac{1}{2}\lambda\phi^2, \tag{A.18}$$

which is just a mass term for the scalar. All this does is modify the diagonal entries of the kinetic term operator so that we now have:

$$\widetilde{\Gamma}_{ij} = \Gamma_{ij} - \lambda\delta_{ij}. \tag{A.19}$$

It turns out that the structure of $\widetilde{\Gamma}$ can be quite sensitive to the choice of perturbation. To see why, we again ask about the eigenvalues of our matrix Γ, as determined by the roots of its characteristic polynomial. For example, reference [302] finds:

$$\det(\lambda\mathbb{I}_{N\times N} - \Gamma) = U_N\left(\frac{\lambda}{2} - 1\right), \tag{A.20}$$

where $U_N(x)$ is a Chebyshev polynomial of the second kind.[98] Restricting to $\lambda \in \mathbb{Z}/N\mathbb{Z}$, we can tabulate when this polynomial vanishes. This gives a sense of how frequently a mass parameter will end up generating a zero mode "by accident." We give the number of zeros for $N = p^a$ for p and a "small numbers":

# zeros	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$	$p=17$
p^1	1	3	3	5	7	9
p^2	1	1	1	1	1	1
p^3	1	3	3	5	7	9
p^4	1	1	1	1	1	1

(A.21)

So, we see that for $N = p$ a prime number, nearly half of the possible deformations produce a zero mode! We note that this pattern appears to also persist for $N = p^a$ for a odd.

A.3 Adding a ϕ^p Potential

To analyze some additional structures in this setting, we now specialize further to the case of $N = p$ a prime number. We can consider adding a perturbation by a potential energy term. While we expect a full analysis may be difficult, there are a few simplifications which occur for specific sorts of perturbations. To illustrate, we now switch on a non-trivial potential:

$$V(\phi) = \lambda\phi^p. \tag{A.22}$$

Now, in characteristic zero, this is a challenging system to study, and a common strategy is to resort to perturbation theory in the parameter λ. In characteristic p, however, note that all elements of $\mathbb{F}_p$ satisfy:

$$\phi^p = \phi \quad \text{for} \quad \phi \in \mathbb{F}_p. \tag{A.23}$$

[98]Recall that the Chebyshev polynomials of the second kind are defined inductively via the following recursion relation: Begin with $U_0(x) = 1$, and $U_1(x) = 2x$. Then, $U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x)$ for $m \geq 1$. For additional details and review, see e.g., [303] as well as [304].

So, assuming that $\phi \in \mathbb{F}_p$, we can simplify this potential to:

$$V(\phi) = \lambda \phi. \tag{A.24}$$

We emphasize that here we are again assuming the "lattice approximation" is in effect; such a simplification is not available if we treat ϕ as a a genuine morphism between schemes.

The generating function for correlation functions is also straightforward to evaluate. Adding a source term as in our previous example, the action is:

$$S = \sum_{i,j} \frac{1}{2} \Gamma_{ij} \phi\,(i)\,\phi(j) + \sum_i (-\lambda \phi(i) + \phi(i) J(i)), \tag{A.25}$$

So if we view the λ's as specifying a constant function $\lambda(i) = \lambda$ for all i, we can complete the square as before:

$$S = \frac{1}{2} \Gamma_{ij} (\phi(i) + \Gamma^{-1}_{ii'} (J(i') - \lambda(i'))(\phi(j) + \Gamma^{-1}_{jj'} (J(j') - \lambda(j')) - \frac{1}{2} \Gamma^{-1}_{ij} (J(i) - \lambda(i))(J(j) - \lambda(j)). \tag{A.26}$$

So in other words, we can just make the substitution $J \mapsto J - \lambda$.

A.4 Another Simple Potential: ϕ^{p+1}

Let us again work in the special case $N = p$. Here we consider a few additional examples of potentials which where we can compute in exact terms the associated correlation functions.

As a first example, consider switching on the non-trivial potential:

$$V(\phi) = \lambda \phi^{p+1}. \tag{A.27}$$

Again assuming that we are working in the lattice approximation so that we can treat ϕ as valued in $\mathbb{F}_p$, we observe that since $\phi^p = \phi$, we have:

$$\phi^{p+1} = \phi^2 \quad \text{for} \quad \phi \in \mathbb{F}_p. \tag{A.28}$$

In particular, this means that adding such a potential interaction amounts to just adding a mass term, a case we already considered in section A.2. A particularly amusing case to consider is $p = 3$, for which we observe that "ϕ^4 theory" collapses to the case of a free field theory. Of course, for more general values of the prime p, no such simplification is available.

A.5 A Deceptively Simple Potential: $\phi^{m(p-1)}$

As a seemingly "trivial" example, consider again the special case $N = p$, but now with the potential:

$$V(\phi) = \lambda \phi^{m(p-1)}, \tag{A.29}$$

where $m \in \mathbb{Z}_{>0}$ is a positive integer. In this case, we observe that for any non-zero $\phi \in \mathbb{F}_p$, we have:

$$\phi^{p-1} = 1 \quad \text{for} \quad \phi \in \mathbb{F}_p^{\times}. \tag{A.30}$$

So in these cases, $V(\phi)$ evaluates to λ. On the other hand, when $\phi = 0$, $V(\phi)$ evaluates to zero. In the end, then, this case turns out to not be entirely trivial, in spite of initial appearances.

B Lattice versus Hasse Derivatives

In this note we have alluded several times to the intuition that we can replace lattice derivatives for physical fields with derivatives of polynomials in characteristic p. In this sense, we can always view the lattice formulation as providing an approximation. We emphasize, however, that the space of polynomials retains further smooth structure which is often absent in lattice field theory. In this Appendix we discuss some further aspects of various notions of derivative in this setting.[99]

For ease of exposition, we focus on the case of a degree M polynomial in a single variable $\phi(t) \in \mathbb{Z}[t]$, and its reduction modulo $N = p^a$ a prime power. For now, we do not restrict the degree of the polynomial $\phi(t)$ so in principle we can allow the polynomial to have degree larger than N. Recall that in the lattice formulation, we consider evaluations of the polynomial at a generic point $x \in \mathbb{Z}/N\mathbb{Z}$, and construct suitable finite differences. We can also see the appearance of derivatives of polynomials via the Taylor expansion:

$$\phi(t+x) = \sum_{r=0}^{m} \mathcal{D}^{(r)}\phi(t) \cdot x^r, \tag{B.1}$$

where $\mathcal{D}^{(r)}$ refers to the rth Hasse derivative,[100] which acts on a monomial t^n with $0 \leq r \leq n$ as:

$$\mathcal{D}^{(r)} t^n = \frac{n!}{r!(n-r)!} t^{n-r}, \tag{B.2}$$

and yields zero when $r > n$. The key point for us is that for each monomial, there is at least one Hasse derivative which is non-zero. Scanning over the different values of x, we get a set of linear relations between the values of $\phi(t+x)$ and the Hasse derivatives:

$$\begin{bmatrix} \phi(t) \\ \phi(t+1) \\ \vdots \\ \phi(t+N-2) \\ \phi(t+N-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1^1 & \dots & 1^{M-1} & 1^M \\ 1 & 2^1 & \dots & 2^{M-1} & 2^M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (N-1)^1 & \dots & (N-1)^{M-1} & (N-1)^M \end{bmatrix} \cdot \begin{bmatrix} \mathcal{D}^{(0)}\phi(t) \\ \mathcal{D}^{(1)}\phi(t) \\ \vdots \\ \mathcal{D}^{(M-1)}\phi(t) \\ \mathcal{D}^{(M)}\phi(t) \end{bmatrix}, \tag{B.3}$$

or more succinctly:

$$\phi(t+j) = \sum_{r=0}^{M} C_{jr} \mathcal{D}^{(r)}\phi(t), \tag{B.4}$$

where C_{jr} are the entries of the $N \times (M+1)$ matrix implicitly defined by equation (B.3). The key point for us is that given the collection of Hasse derivatives, we can finite differences, as

[99] Again, compared with standard lattice field theory, here we are discretizing both the source and target spaces.

[100] See e.g., [305].

would be associated with the lattice derivative. For example, the first lattice derivative at t is given by:

$$D_{\text{lat}}\phi(t) \equiv \phi(t+1) - \phi(t), \tag{B.5}$$

and the second lattice derivative at t is given by:[101]

$$D^2_{\text{lat}}\phi(t) \equiv \phi(t+1) - 2\phi(t) + \phi(t-1). \tag{B.6}$$

Similar considerations hold for the higher lattice derivatives. The important point for us is that these lattice derivatives always make reference to a finite set. Indeed, strictly speaking when we discuss lattice derivatives, t is no longer a formal variable, but is to be viewed as being evaluated at a specific point of $\mathbb{Z}/N\mathbb{Z}$.

Turning the discussion around, we can also ask whether we can start from a collection of finite evaluations and reconstruct the full set of Hasse derivatives, and thus implicitly the full polynomial $\phi(t)$. If $M \geq N$, then there are in general more Hasse derivatives than finite differences. This illustrates that beyond a certain point, our lattice approximation will not work, but the formulation of physical fields as morphisms will still apply.

If, however, $M = N$, there is a chance that we can still reconstruct all the available Hasse derivatives directly from finite differences. As we now explain, this works when $N = p$, but for $N = p^a$ for $a > 1$, more care is needed since we cannot take inverses in $(\mathbb{Z}/N\mathbb{Z})^\times$, the multiplicative group of integers modulo N.

To establish this, we observe that C is an example of a Vandermonde matrix :

$$\text{Vand}(x_1, ..., x_N) = \begin{bmatrix} 1 & x_0 & x_0^2 & ... & x_0^M \\ 1 & x_1 & x_1^2 & ... & x_1^M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_M & x_N^2 & ... & x_M^M \end{bmatrix}. \tag{B.7}$$

and in our case, we have $x_j = j$ for $j = 0, ..., M$. Now, the determinant of a general Vandermonde matrix is given by:

$$\det \text{Vand}(x_1, ..., x_N) = \prod_{0 \leq i < j \leq M} (x_j - x_i). \tag{B.8}$$

[101] It would of course have been more pleasant to define a lattice difference via the more symmetric expression "$D\phi(t) = \phi(t+1/2) - \phi(t-1/2)$" since one could then consistently iterate to produce the higher derivatives, e.g., "$D^2\phi(t) = D\phi(t+1/2) - D\phi(t-1/2) = \phi(t+1) - 2\phi(t) + \phi(t-1)$". Unfortunately, while "1/2" does indeed make sense when working over $(\mathbb{Z}/N\mathbb{Z})^\times$ for N relatively prime to 2, the meaning of 1/2 is rather different than in the case where we equip $\mathbb{Q}$ with the standard Euclidean norm. Indeed, working in $(\mathbb{Z}/N\mathbb{Z})^\times$ we have the integer representative $1/2 = (N+1)/2$, which is "halfway around the world". For these reasons, we have opted to make do with this cruder set of definitions which adhere to more standard lattice considerations.

So in our case, we obtain:

$$\det C = \prod_{0 \le i < j \le M} (j - i), \tag{B.9}$$

or, expanding out the product, we can instead write:

$$\det C = \prod_{m=2}^{M} m^{M+1-m}. \tag{B.10}$$

We now observe that in this product, all the individual entries are non-zero modulo N. In particular, this means that for $N = p$ a prime, the product reduces modulo p to an element of $\mathbb{F}_p^\times$, so an inverse does indeed exist. We can then invert the matrix C and extract the appropriate finite differences.

But when $N = p^a$, the same reasoning shows that the inverse of $\det C$ does not exist in $\mathbb{Z}/N\mathbb{Z}$. Indeed, recall that $(\mathbb{Z}/N\mathbb{Z})^\times$, the multiplicative group of integers modulo N is specified by integers in the set $\{1, ..., N-1\}$ which are relatively prime to N. In our product formula for $\det C$, we observe that every single integer between 1 and $N-1$ appears in the product, so unless N is prime, it will not have a well-defined inverse in $\left|(\mathbb{Z}/N\mathbb{Z})^\times\right|$, and so we cannot view C^{-1} as a matrix with entries in $\mathbb{Z}/N\mathbb{Z}$. Of course, we can compute the inverse of C over $\mathbb{Q}$, and then clear denominators, though this is also not without its own difficulties. This provides a possible workaround to reconstructing all the Hasse derivatives, but it illustrates that additional care is needed in such cases.

Similar considerations clearly apply if we work with N a more general positive integer.

C Finite Fields

In this Appendix we briefly review some aspects of finite fields. We cannot hope to provide a full review of this material, and so instead refer the interested reader to an abstract algebra textbook for further details, for example [306, 307].

To begin, we recall that in abstract algebra, a field K has both a commutative addition and multiplication operations such that its elements form a group under addition, and after deleting 0, the identity of the additive group law, the remaining elements $K^\times$ form a multiplicative group with identity 1. Common examples include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. More abstractly, one can consider fields such as $\mathbb{Q}(t)$, $\mathbb{R}(t)$, $\mathbb{C}(t)$, with elements given by ratios of polynomials in a formal variable t. All of these examples have an infinite number of elements and specify characteristic zero fields.

One can also construct finite fields by observing that $\mathbb{Z}/p\mathbb{Z}$, the integers modulo p a prime number also satisfies all the requirements to be an algebraic field. This field is denoted as $\mathbb{F}_p$. We will shortly introduce additional finite fields $\mathbb{F}_q$ with $q = p^n$. The most important feature of all these fields is that they have characteristic p, meaning $p = 0$ in the field. Another important consequence for any characteristic p field is that we have the "Freshman's dream" equation:

$$(x + y)^p = x^p + y^p \tag{C.1}$$

This follows from expanding out the polynomial and observing that all but two coefficients are equal to zero modulo p. We also have "Fermat's little theorem" which tells us that for $m \in \mathbb{Z}$:

$$m^p \equiv m \bmod p, \tag{C.2}$$

The Frobenius map is defined by taking elements of a ring R (and thus also a field) and multiplying p times:

$$F : R \to R \tag{C.3}$$

$$r \mapsto r^p. \tag{C.4}$$

Note that for the field $\mathbb{F}_p$, all elements are fixed under this map. Consequently, we can speak of the Frobenius field endomorphism (namely one which respects addition and multiplication of the field):

$$F : \mathbb{F}_p \to \mathbb{F}_p. \tag{C.5}$$

We now introduce the finite fields $\mathbb{F}_q$. In the spirit of Galois theory, we look for the roots of irreducible polynomials over a field K. Adjoining these solutions to our original field, we obtain a field extension L, which we can view as a vector space with coefficients in K. For example, $\mathbb{C} = \mathbb{R}(i)$ where $i^2 + 1 = 0$.

Given an irreducible degree n polynomial $P_n(t)$ in the ring $\mathbb{F}_p[t]$, solving the equation:

$$P_n(t) = 0 \tag{C.6}$$

will produce a field extension of $\mathbb{F}_p$ when $n > 1$. Indeed, the condition that $P_n(t)$ is irreducible means that we can build a bigger field by adjoining the roots of $P_n(t)$ to $\mathbb{F}_p$. Denoting one such root by α so that $P_n(\alpha) = 0$, observe that $F(\alpha) = \alpha^p$ is also a root, since:

$$P_n(\alpha^p) = (P_n(\alpha))^p = 0, \tag{C.7}$$

where we used the Freshman's dream. Note also that only elements of $\mathbb{F}_p$ are fixed under Frobenius conjugation, so $F(\alpha)$ is distinct from α. Indeed, it turns out the Frobenius map generates the Galois group $\mathrm{Gal}\left(\mathbb{F}_p(\alpha)/\mathbb{F}_p\right) \simeq \mathbb{Z}/n\mathbb{Z}$, the cyclic group with n elements. Viewed as a vector space, we can treat elements of this new field as n-component vectors. But since each component has p possible entries, the total number of possible entries is p^n. This is the number of elements in the finite field $\mathbb{F}_q$ with $q = p^n$. We can again ask how the Frobenius endomorphism acts on this field. In this case, it turns out that only elements of $\mathbb{F}_p$ remain invariant. It is convenient to work in terms of a basis spanned by the images of our root α under the Frobenius map, so we can write a general element $y \in \mathbb{F}_q$ as:[102]

$$y = y_0\alpha + y_1\alpha^p + ... + y_{n-1}F^{n-1}(\alpha), \tag{C.8}$$

where the $y_j \in \mathbb{F}_p$ for $j = 0, ..., n-1$. Observe that any element of this bigger field satisfies the equation:

$$y^q = y, \tag{C.9}$$

which follows from the fact that F has order n on $\mathbb{F}_q$.

Continuing in this way, we can construct field extensions of $\mathbb{F}_q$ as well. We denote by $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$. We note that this field also has characteristic p, but it clearly has an infinite number of elements. In this case, the Frobenius automorphism also has infinite order. One can also construct infinite order fields in characteristic p such as $\mathbb{F}_p(t)$, or more generally, the field of functions for a variety.

One feature we have used in our analysis is that we can work with V a dimension n vector space over $\mathbb{F}_p$ and interpret the action of a "dot product" in terms of an algebraic operation on $\mathbb{F}_q$. To see how this works in detail, fix a choice of a non-degenerate symmetric bilinear

[102]More generally, we can invoke the normal basis theorem of Galois theory, which asserts that for a Galois extension L/K, there exists an element $\beta \in L$ such that $\{g(\beta) | g \in \mathrm{Gal(L/K)}\}$ forms a basis of L when viewed as a vector space over K. To give a perhaps more familiar example, consider $\mathbb{C} = \mathbb{R}(\sqrt{-1})$ as a quadratic extension of $\mathbb{R}$, where the Galois group acts by $\sqrt{-1} \mapsto -\sqrt{-1}$. Then, the β in question is $\beta = 1 + \sqrt{-1}$, since its conjugate is $1 - \sqrt{-1}$. Observe also that in contrast to the characteristic p case one cannot simply take β to be $\sqrt{-1}$, since that would not span the vector space.

form:[103]

$$B : V \times V \to \mathbb{F}_p \tag{C.10}$$
$$(v, w) \mapsto B^{ij} v_i w_j. \tag{C.11}$$

Note that for any element $y \in \mathbb{F}_q$, we have an expansion in terms of powers of α as well as its Frobenius conjugates:

$$y = y_0 \alpha + y_1 \alpha^p + ... + y_{n-1} F^{n-1}(\alpha) \tag{C.12}$$
$$F(y) = y_{n-1} \alpha + y_0 \alpha^p + ... + y_{n-2} F^{n-1}(\alpha) \tag{C.13}$$
$$... \tag{C.14}$$
$$F^{n-1}(y) = y_1 \alpha + y_2 \alpha^p + ... + y_0 F^{n-1}(\alpha), \tag{C.15}$$

so by linear algebra on $\mathbb{F}_q$, we can invert this relation to write:

$$y_i = M_{ij} F^j(y), \tag{C.16}$$

for a matrix M_{ij} determined only by α. Now we can rewrite our "dot product" as a pairing:

$$B : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p \tag{C.17}$$
$$(v, w) \mapsto B^{ij} M_{ii'} F^{i'}(v) M_{jj'} F^{j'}(w), \tag{C.18}$$

where we have abused notation in treating v and w as elements of $\mathbb{F}_q$.

[103] Here we are not using the physicist convention for upper and lower indices. We do this to avoid confusion with raising a given element to some power.

D Geometry in Characteristic p

In this Appendix we briefly discuss some aspects of geometry in characteristic p. We refer the interested reader to standard texts in algebraic geometry such as [106] for further details. Our discussion will also follow the lectures [308].

To set the stage, we recall that in algebraic geometry, we first specify a commutative ring R, and then build up an affine patch of the geometry from $\operatorname{Spec} R$, the set of all prime ideals in R.[104] To illustrate, the affine complex line $\mathbb{A}^1$ can be thought of as $\operatorname{Spec} \mathbb{C}[x]$. Indeed, the prime ideals of $\mathbb{C}[x]$ are generated by polynomials of the form $(x - c)$ for $c \in \mathbb{C}$. Each of these values of c specifies a point on our affine line. As a somewhat less intuitive example, one can even consider $\operatorname{Spec} \mathbb{Z}$ which consists of the ideals generated by the prime integers, as well as the element 0.

There is also a notion of localizing at at a given element of $\operatorname{Spec} R$. Given a prime ideal $\mathfrak{p}$, we define $R_{\mathfrak{p}}$ by first constructing the complement $\mathfrak{p}^c = R \backslash \mathfrak{p}$. Then, we are free to take inverses of $\mathfrak{p}^c$ inside R, building a new ring:

$$R_{\mathfrak{p}} = (\mathfrak{p}^c)^{-1} R. \tag{D.1}$$

One can think of this as allowing us to build fractions from objects inside R.

We are now ready to construct the cotangent space. Our discussion follows the notes of reference [308]. Given a prime ideal $\mathfrak{p} \subset R$, we get a point $[\mathfrak{p}] \in \operatorname{Spec} R$. We can then construct $R_{\mathfrak{p}}$, the localization of the ring at this point. This new ring has a maximal prime ideal $\mathfrak{p} R_{\mathfrak{p}} = \mathfrak{m}$. Observe that $[\mathfrak{p} R_{\mathfrak{p}}]$ is a point of $\operatorname{Spec} R_{\mathfrak{p}}$. From this data, we can construct the residue field:

$$k = R_{\mathfrak{p}} / \mathfrak{m}, \tag{D.2}$$

as well as a vector space $V = \mathfrak{m}/\mathfrak{m}^2$ over the field k. The vector space V is the Zariski cotangent space at $[\mathfrak{p}]$, and we write $T_x^* X$ to denote the cotangent space of a scheme X at a point x.

This notion of cotangent space is actually quite flexible. As a particularly counter-intuitive example, we can consider $\operatorname{Spec} \mathbb{Z}$ and calculate the derivative of integers at different primes. For example, given $40 = 2^3 \times 5$, we see that it vanishes at both the point $[2]$ and the point $[5]$. Computing the derivatives at these two points yields:

$$\frac{d}{d[2]} 40 = 3 \times 2^2 \times 5 = 0 \bmod 2 \tag{D.3}$$

$$\frac{d}{d[5]} 40 = 2^3 = 3 \bmod 5. \tag{D.4}$$

[104] Recall that an ideal $I \subset R$ is defined by the properties that as an additive group, it is a subgroup of R, and that for $r \in R$ and $m \in I$, $rm \in I$. A prime ideal P is one for which if $a, b \in R$ and $ab \in P$, then either a or b is an element of P.

Our discussion so far has focused on notions from "classical" algebraic geometry. This is enough for us to start equipping our space with appropriate sheaves, and a notion of local structure.

Even in characteristic p, there is a notion of a local analytic isomorphism as associated with a diffeomorphism. These are known as étale morphisms [156]. There are several equivalent ways to phrase this condition more precisely. We refer to a morphism of schemes $f : X \to Y$ as étale if it satisfies the condition that f is flat,[105] locally of finite presentation,[106] and for every $y \in Y$, the fiber $f^{-1}(y)$ is the disjoint union of points, each of which is the spectrum of a finite separable field extension of the residue field $\kappa(y)$. We found the entry [311] helpful in providing additional definitions.

Giving a full treatment would carry us to far afield from our main developments, but there is one important "moral point" to emphasize. Perhaps the most important observation is that in characteristic zero, the notion of an étale map matches up well with the condition of a map being analytic. Closely following the discussion in [312], for a map $\phi : X \to Y$ of locally finite type $\mathbb{C}$-schemes, the associated map of complex-analytic spaces $\phi^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local isomorphism if and only if ϕ is étale. In characteristic p, the main issue is in specifying the analog of the inverse function theorem.

More generally, we can speak of smooth morphisms of schemes $f : X \to Y$. A morphism f is smooth provided it is flat, finitely presented, and specified by the condition that for all $y \in Y$, $f^{-1}(y)$ is a smooth scheme over the residue fields $\kappa(y)$. One can also view smooth morphisms $f : X \to Y$ as defined by the condition that locally, they factor through an étale map $X \xrightarrow{g} \mathbb{A}^n_S \to Y$, where here $\mathbb{A}^n_S$ is affine n-space over a scheme S. In characteristic zero differential geometry, the smooth morphisms specify smooth submersions.

Curves in Characteristic p

We now briefly discuss curves in characteristic p. As we already mentioned, one choice is to just take $\mathbb{F}_q = \mathbb{A}^1$, the affine line. This clearly has q distinct points. One can also consider the affine line given by the zero set of the equation:

$$x + y = 0 \tag{D.5}$$

for $x, y \in \mathbb{F}_q$. Again, this is an affine line and one can verify that this also has precisely q points. More generally, we can consider cutting out a one-dimensional subspace from a

[105]Essentially quoting from [309], recall that an R-module M is called flat if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules, the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is also exact. A ring map $R \to S$ is called flat if S is flat as an R-module. Continuing in this way, we can specify a flat morphism by its actions on the underlying defining rings.

[106]Namely, if we consider local neighborhoods U of X and $V \supset f(U)$ of Y, then $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra. For additional discussion see for example [310].

hypersurface equation such as:

$$f(x,y) = \sum_{i,j} f_{ij} x^i y^j = 0, \tag{D.6}$$

for $f(x,y) \in \mathbb{F}_q[x,y]$ a polynomial in two variables. Even more generally, we can add more coordinates and consider additional intersections of hypersurfaces. Observe that as we do this, the number of points in the ambient geometry, namely $\mathbb{F}_q^m$ also grows, becoming q^m in order. This illustrates that even in characteristic p, discretization need not mean that we are stuck with just the affine line.

It is often easier to work with hypersurfaces in a projective space. For example, for a curve in a projective space $\mathbb{P}^2$, we can write it as the zero set of the equation:

$$\left\{ f(x,y,z) = \sum_{i,j,k} f_{ijk} x^i y^j z^k = 0 \right\} \subset \mathbb{P}^2, \tag{D.7}$$

where $f(x,y,z)$ is a homogeneous polynomial in three variables.

The notion of a genus can be specified in much the same way as in characteristic zero. Indeed, for a projective curve Σ we can introduce the canonical sheaf $\mathcal{K}_\Sigma$ and then use the Riemann-Roch theorem to calculate the genus:

$$h^0(\mathcal{K}_\Sigma) - h^1(\mathcal{K}_\Sigma) = 2g - 2. \tag{D.8}$$

As a simple example, note that a plane curve of degree d has genus $g = (d-1)(d-2)/2$. In most well-behaved situations with a polynomial with integer coefficients reduced modulo p, this genus behaves just like its characteristic zero counterpart, though there are some notable exceptions. As a pathological example, note that the equation $y = x^q + x = 2x$ for $x, y \in \mathbb{F}_q$.

One can also ask about the number of points in this curve. For a curve Σ defined over $\mathbb{F}_q$, there is also an important Hasse-Weil bound on the number of solutions (see e.g., [24]):

$$|\#\Sigma(\mathbb{F}_q) - (q+1)| \leq 2g\sqrt{q}. \tag{D.9}$$

E Grothendieck Topologies

In this Appendix we briefly review some aspects of Grothendieck topologies. We refer the interested reader to [313] for further elaboration on the subject. For our purposes, the main point of these notions is to provide a suitable generalization of covering spaces which can produce a non-trivial cohomology theory, even in the discretized situation present in defining varieties over finite fields. Essentially quoting from [313], one defines a topology or site T as a category $\mathrm{cat}(T)$ of a set of coverings $\mathrm{cov}(T)$ defined as families of morphisms $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ in $\mathrm{cat}(T)$ such that the following three properties hold:

- (T1) For $\{U_i \to U\}$ in $\mathrm{cov}(T)$ and a morphism $V \to U$ in $\mathrm{cat}(T)$, all fiber products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}$ is again in $\mathrm{cov}(T)$.
- (T2) Given $\{U_i \to U\} \in \mathrm{cov}(T)$, and a family $\{V_{ij} \to U_i\} \in \mathrm{cov}(T)$ for all $i \in I$, the family $\{V_{ij} \to U\}$ obtained by composition of morphisms also belongs to $\mathrm{cov}(T)$.
- (T3) If $\varphi : U' \to U$ is an isomorphism in $\mathrm{cat}(T)$ then $\{U' \xrightarrow{\varphi} U\} \in \mathrm{cov}(T)$.

Again, the point of these notions is to have a sense of open coverings as one has in standard topology, but in which the emphasis is on the morphisms rather than the sets themselves. Standard notions of presheaves and sheaves can be defined in this setting as well. For example, letting $\mathcal{C}$ denote a category of products (which can include the case of the category of abelian groups or the category of sets), and T a topology, we can define a presheaf on T with values in $\mathcal{C}$ as a contravariant functor $F : T \to \mathcal{C}$. We can then speak of a sheaf on T as defined by the condition that if for every covering $\{U_i \to U\}$ in T, the following diagram is exact:

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j), \tag{E.1}$$

namely the diagram of line (E.1) is an equalizer.[107] In the above, $U_i \times_U U_j$ is the standard fiber product as specified by the commutative diagram:

$$\begin{tikzcd} U_i \times_U U_j \arrow[r] \arrow[d] & U_j \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd} \tag{E.2}$$

where the structure of the exact sequence of line (E.1) now follows since F is a contravariant functor.

[107] Recall that for sets X and Y and maps $f : X \to Y$ and $g : X \to Y$, the equalizer $\mathrm{Eq}(f,g) = \{x \in X \text{ such that } f(x) = g(x)\}$. One can extend this in the obvious way to any collection of maps from X to Y.

The main usage of this formalism for us is in defining the various étale topologies. In particular, given a scheme X, we can specify the category of étale X schemes, denoted by $\text{Ét}/X$. The (small) étale site (i.e. topology) of X is denote by $X_{\text{ét}}$, where $\text{cat}(X_{\text{ét}})$ is just $\text{Ét}/X$ and the space of coverings $\text{cov}(X_{\text{ét}})$ is the set of surjective families of morphisms in $\text{Ét}/X$. One can also speak of a big étale site, but at the level of cohomology, these distinctions are often not important, and we will not elaborate on this further.

The notion of a crystalline site also implicitly appears in our discussion. From [314], if X is a scheme over a field k, then the crystalline site of X relative to W_n, denoted $Cris(X/W_n)$, has as its objects pairs $U \to T$ consisting of a closed immersion of a Zariski open subset U of X into some W_n-scheme T defined by a sheaf of ideals J, together with a divided power structure on J compatible with the one on W_n.

Given a suitable notion of global sections for sheaves, we can construct the associated cohomology theory via right-derived functors of the global sections. Given a sheaf F and open cover $\{U_i \to U\}_{i\in I}$, we consider the left-exact functor $F \mapsto H^0(\{U_i \to U\}_{i\in I}, F)$. Then, the right-derived functor provides a definition of the higher degree cohomology groups: $H^j(U, \mathcal{F}) = R^j(U, \mathcal{F})$ (see e.g., [313]). Using the sheaf property, we can then extend to X.

For our purposes, the utility of introducing the étale topology is that we also have the Artin comparison theorem [315], which states, for an algebra A given for example by either a finite field $\mathbb{F}_q$, the p-adic ring of integers $\mathbb{Z}_p$, or the p-adic numbers $\mathbb{Q}_p$ (and suitable generalizations thereof) that:

$$H^\bullet(X_{\text{ét}}, A) \simeq H^\bullet(\mathcal{X}^{\text{an}}, A), \tag{E.3}$$

where $\mathcal{X}^{\text{an}}$ refers to the analytification of X over $\mathbb{C}$, i.e., we interpret our variety as defined over $\mathbb{C}$ and then equip it with the standard topology of an analytic space.

F Codes

In this Appendix we review some aspects of classical and quantum codes alluded to earlier. For a review of linear and non-linear codes, see for example the thesis [316]. For a review of algebraic geometry codes, see for example reference [317]. For a review of how quantum codes can be obtained from algebraic codes, see reference [318]. Our plan will be to first review some aspects of classical coding theory, and in particular the relation to algebraic curves over finite fields. We then turn to quantum codes obtained from these classical codes. As throughout, we let q denote a power of some prime p.

F.1 Classical Algebraic Codes

To set the stage, let us recall that the main idea in much of classical coding theory is to send messages over a noisy channel. Our discussion will follow that given in reference [316]. More precisely, one has in mind the following schematic diagram:

$$\underbrace{[\text{Source}] \to [\text{Transmitter}]}_{\text{Input}} \to \underbrace{[\text{Receiver}] \to [\text{Sink}]}_{\text{Output}}. \tag{F.1}$$

The "source" and "sink" may consist of k different possible messages which are then encoded in a larger set of n "codewords" for the transmitter and receiver.[108] Denote by V the set of input words and by C the set of possible codewords. A passed message will be denoted as $C(v)$ for $v \in V$.

The main idea is that by a suitable embedding of k possible messages in the codewords, random errors in the transmission can be minimized. Of course, one way to proceed is to encode all information in a string of 1's and 0's, but more generally, our basic alphabet may consist of a q-ary code with q different possible letters, as for example would occur if we use the finite field $\mathbb{F}_q$. A code is then some collection of different codewords of length n.

Now, in passing a message from the transmitter to the receiver, there may be some noise, i.e., errors may be generated. This amounts to flipping some of the entries in our codeword. Much of the art of the subject revolves around finding efficient ways to protect messages so that even when errors are present, the message can be decoded. Along these lines, We refer to the redundancy of a code as $n - k$ and the information rate of a code as:

$$R = \frac{1}{n} \log_q |C| \,. \tag{F.2}$$

We can speak of the weight of a codeword as the number of entries which are different from zero. Given a code C, we can also specify the distance d by computing the Hamming distance of elements in the image set $C(V)$ as specified by taking an input word in V and encoding

[108] Clearly, here k refers to a positive integer rather than a field. We have begun to exhaust the limits of the English alphabet.

it in C:

$$d(C) = \min_{v \neq v'} \{\mathrm{dist}_{\mathrm{Ham}}(C(v), C(v')) \text{ with } v, v' \in V\}\,, \tag{F.3}$$

where $\mathrm{dist}_{\mathrm{Ham}}$ is the "Hamming distance", i.e., we view our words as elements of $\mathbb{F}_q^n$ with respect to a fixed basis and count the number of entries over which the two vectors are different. When the context is clear, we shall often just write C to denote the code space.

Specializing further, we can build a large set of codes by assuming the various words and codes are built from vectors in vector spaces over $\mathbb{F}_q$. In this case, the source is just a k-dimensional vector space $V \simeq \mathbb{F}_q^k$ and the code space is an n-dimensional vector space $W \simeq \mathbb{F}_q^n$. We can speak of the embedding $V \to W$ as specifying a code C, i.e., it is just the image set of V inside W. In this case, we can view errors as elements $e \in W$ so that for a codeword $C(v) \in W$, the error is just given by $C(v) + e$, i.e., it flips some of the entries.

Given a linear code C, we can also specify the distance d by computing the Hamming distance of elements in the image set $C(V)$, just as we did in line (F.3). Note that $d(C)$ does not depend on this choice of basis since we always minimize over all vectors in the image anyway. We refer to a linear $[n, k, d]_q$ code where $d(C) = d$ is the minimum distance of the code. In this case, the information rate is just $R = k/n$.

One can also introduce a notion of non-linear codes. Treating each codeword as an element of $\mathbb{F}_q^n$ with respect to a fixed basis, we can view this as defining a set $C_{\mathrm{nl}} \subset \mathbb{F}_q^n$, where the subscript serves to remind us that this is not a vector space. We refer to the kernel of this space as $K(C_{\mathrm{nl}})$:

$$K(C_{\mathrm{nl}}) = \{v \in C_{\mathrm{nl}} \text{ with } \lambda v + C_{\mathrm{nl}} = C_{\mathrm{nl}} \text{ for all } \lambda \in \mathbb{F}_q\}\,, \tag{F.4}$$

which is a vector space. The rest of the codewords can then be obtained by adding appropriate vectors, i.e., by suitable affine transformations:

$$C_{\mathrm{nl}} = \bigcup_{i=1}^{t} (K(C_{\mathrm{nl}}) + v_i)\,, \tag{F.5}$$

that is, we introduce t coset vectors $v_1, ..., v_t$ to build the rest of the codewords.

Algebraic varieties over finite fields provide a way to build examples of codes, a topic we now review following [317]. Intriguingly, the additional geometric structure present in this class of examples often provide a way to build "good" examples in the sense that certain information theoretic quantities can be handled analytically. We cannot hope to provide a full characterization of the subject, but we can at least explain how these geometric ingredients emerge.

To keep things as concrete as possible, we fix X a smooth, projective irreducible curve of genus g over the finite field $\mathbb{F}_q$. We begin by introducing two sets of points which we write as $P_1, ..., P_n$ and $Q_1, ..., Q_m$. From these, we can form the divisors $D = P_1 + ... + P_n$ and $G = Q_1 + ... + Q_m$. We can then introduce the Riemann-Roch vector space associated with this divisor:

$$\mathcal{L}(G) = \{f \in \mathbb{F}_q(X) \quad \text{such that} \quad (f) + G \geq 0\} \cup \{0\}\,. \tag{F.6}$$

At this point, it is helpful to recall that for a plane curve in $\mathbb{P}^2$ defined by the equation $h(x, y, z) = 0$ with x, y, z homogeneous coordinates, the space of functions is given by ratios of the form $p(x, y, z)/q(x, y, z)$ where h does not divide either p or q. This property ensures that $\mathcal{L}(G)$ is finite dimensional.

We note that $\mathcal{L}(G)$ is also just the space of global sections for a line bundle, and in this case it is customary to denote it as $H^0(X, \mathcal{O}_X(G))$. This defines a vector space which we denote by $\ell(G) = k$. We can introduce a basis which we write as $\{f_a\}$ for $a = 1, .., k$. Given these functions, we can produce a code by evaluating at all n points of D:

$$\mathrm{ev}_D : \mathcal{L}(G) \to \mathbb{F}_q^n \tag{F.7}$$

$$f \mapsto (f(P_1), ..., f(P_n)). \tag{F.8}$$

Doing so, we get an $n \times k$ matrix as specified by $f_a(P_i)$ where $a = 1, ..., k$ and $i = 1, ..., n$. The resulting code is often denoted as $C_{\mathcal{L}}(D, G)$. We note that if $\deg G < n$, then this specifies an $[n, k, d]_q$ linear code with n set by the number of evaluation points, $k = l(G)$ the dimension of the linear system, and $d = n - \deg G$ the minimal distance between codewords.

In the literature on algebraic geometry codes, it is also customary to discuss the space of meromorphic one forms:

$$\Omega(G - D) = \{\omega \in \Omega(X) \quad \text{such that} \quad (\omega) \geq G - D\} \cup \{0\}\,. \tag{F.9}$$

In this case, one specifies a code by computing the residues of ω at the marked points. In other words, the evaluation map in this case is given by:

$$\mathrm{ev}_D : \Omega(G - D) \to \mathbb{F}_q^n \tag{F.10}$$

$$\omega \mapsto (\mathrm{res}_{P_1}\omega, ..., \mathrm{res}_{P_n}\omega), \tag{F.11}$$

and the corresponding linear code is denoted by $C_\Omega(D, G)$. Let us note that there is a duality between the codes $C_{\mathcal{L}}(D, G)$ and $C_\Omega(D, G)$ which is often written as:

$$C_{\mathcal{L}}(D, G) = C_\Omega(D, G)^\perp, \tag{F.12}$$

where we have implicitly used a notion of orthogonality as induced by introducing a pairing on $\mathbb{F}_q^n$ given by:

$$\mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q \tag{F.13}$$

$$(a, b) \mapsto a \cdot b = \sum_{i=1}^{n} a_i b_i, \tag{F.14}$$

and for a vector space $V \subset \mathbb{F}_q^n$, we define

$$V^{\perp} = \left\{ w \in \mathbb{F}_q^n \quad \text{such that } w \cdot v = 0 \quad \text{for all } v \in V \right\}. \tag{F.15}$$

Now we can see why equation (F.12) is true; We can consider any $f \in \mathcal{L}(G)$ and any $\omega \in \Omega(G)$. The dot product between the two is, in the obvious abuse of notation:

$$(f, \omega) = \sum_{i=1}^{n} f(P_i) \operatorname{res}_{P_i} \omega = \sum_{i=1}^{n} \operatorname{res}_{P_i} f\omega = 0, \tag{F.16}$$

where the last equality follows from the fact that we are summing over *all* the residues of a compact curve.

Though we will not be too concerned with "practical applications," some important properties of algebraic codes include the fact that infinite families of codes $[n_i, k_i, d_i]_q$ can be constructed such that the information rate $R_i = k_i/n_i$ and relative distance $\delta_i = d_i/n_i$ remain finite and non-zero as $i \to \infty$.

The above considerations can also be extended to produce a class of non-linear codes which again have good asymptotic coding properties. Following [319] we next consider a stable vector bundle E over the curve X. Stability is defined in essentially the same way as in characteristic zero; we first change base to the algebraic closure $\overline{\mathbb{F}}_q$ and then specify the slope as $\mu(E) = \deg(E)/\operatorname{rk}(E)$, where $\deg(E)$ denotes the degree and $\operatorname{rk}(E) \equiv r$ the rank of the vector bundle. We refer to a bundle as being stable if and only if, for every E' a subbundle of E, we have $\mu(E') < \mu(E)$. Now, the important point for us is that for any point $x \in X$, we can consider the stalk E_x which is just a copy of $\mathbb{F}_q^r \simeq \mathbb{F}_Q$, with $Q = q^r$. We can then proceed much as we did in the line bundle case: we simply consider the evaluation map at n different points of the global sections of E:

$$\text{ev} : H^0(X, E) \to \bigoplus_{i=1}^{n} E_{P_i} \simeq \mathbb{F}_Q^n \tag{F.17}$$

$$v \mapsto (v(P_1), ..., v(P_n)). \tag{F.18}$$

Observe that this need not define an $\mathbb{F}_Q$ linear code, but we do get an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_Q^n$. Note that if the evaluation map is injective, then the size of the code is $K = q^{h^0(X,E)}$, the dimension of the non-linear code is $k = \log_Q K$, and since our code space is $\mathbb{F}_q$-linear, d is just the minimal weight of a non-zero codeword.

Specifying codes is in some sense the "easier" part of the story. Indeed, the utility of a given code also requires one to be able to efficiently decode a given signal, and there is again a vast literature centered around how to do this. We will not dwell on this point since it is beyond the scope of the present considerations.

F.2 Quantum Error Correcting Codes

Let us now turn to the case of quantum error correcting codes generated from algebraic codes. Our discussion follows reference [318]. To begin, we recall that a q-ary quantum code of length n is specified as a dimension k complex subspace $Q \subset (\mathbb{C}^q)^{\otimes n}$. We can adopt a basis of states $|u_1, ..., u_n\rangle$, where each $u_i \in \mathbb{F}_q$ specifies a single qudit register. We can introduce a set of unitary operators E_i which serve to encode possible errors made in transmitting information. There is then an orthogonal decomposition of $(\mathbb{C}^q)^{\otimes n}$ as:

$$(\mathbb{C}^q)^{\otimes n} \simeq \bigoplus_{i=0}^{t} E_i Q, \tag{F.19}$$

where E_0 is the identity, and $t = q^{n-k} - 1$.

Given a quantum code Q with basis $|\psi_1\rangle, ..., |\psi_k\rangle$, we have a notion of errors E and F being "correctable" if they are distinguishable, namely if the following conditions are met:

$$\left\langle \psi_i | E^\dagger F | \psi_j \right\rangle = 0 \quad \text{and} \quad \left\langle \psi_i | E^\dagger F | \psi_i \right\rangle = \left\langle \psi_j | E^\dagger F | \psi_j \right\rangle. \tag{F.20}$$

for all i, j.

A basis of error operators can be specified as follows. Begin by considering the special case where $n = 1$. Then, we note that for some $m \geq 1$, we have $q = p^m$ and $\mathbb{F}_q$ is a vector space over $\mathbb{F}_p$. Introduce $a, b, u \in \mathbb{F}_q$. Given a state $|u\rangle \in \mathbb{C}^q$, we have the qudit operations:

$$T_a |u\rangle = |u + a\rangle \quad \text{and} \quad R_b |u\rangle = \xi^{\mathrm{Tr}(bu)} |u\rangle, \tag{F.21}$$

where ξ is a primitive pth root of unity. Here, we have again used the trace map Tr: $\mathbb{F}_q \to \mathbb{F}_p$ to ensure that all phases have unit modulus. We can then define an "error operation" $E_{ab} = T_a R_b$, and the span of these operators constitute the set of errors on a single qudit.

Next consider the case $n > 1$. By abuse of notation we let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and $u = (u_1, ..., u_n)$. Then, we can introduce error operators:

$$E_{ab} = T_a R_b \tag{F.22}$$

where:

$$T_a = \underbrace{T_{a_1} \otimes ... \otimes T_{a_n}}_{n} \quad \text{and} \quad R_b = \underbrace{R_{b_1} \otimes ... \otimes R_{b_n}}_{n}, \tag{F.23}$$

and we have the error basis:

$$\mathcal{E}_n = \left\{ E_{ab} = T_a R_b \quad \text{with} \quad a, b \in \mathbb{F}_q^n \right\}, \tag{F.24}$$

and the error group:

$$\mathcal{G}_n = \left\{ \xi^i E_{ab} \quad \text{with} \quad a, b \in \mathbb{F}_q^n \quad \text{and} \quad 0 \leq i \leq q-1 \right\}. \tag{F.25}$$

We can also speak of the weight of an error $\xi^i E_{ab}$ as the number of operations different from the identity, i.e.:

$$\text{wt}(\xi^i E_{ab}) = n - |\{i : a_i = b_i = 0\}|, \tag{F.26}$$

and thus, can speak of the minimum distance of a quantum code Q as the maximum weight of error operations which can be corrected

$$d(Q) = \max_{\substack{|u\rangle, |v\rangle \in C \\ E \in \mathcal{G}_n}} \{d \text{ such that } \langle u|v\rangle = 0 \text{ and } \text{wt}(E) \leq d-1 \Rightarrow \langle u|E|v\rangle = 0\}. \tag{F.27}$$

We can now refer to a quantum code of length n, dimension k and minimum distance d as an $[[n, k, d]]_q$ code.

F.2.1 Stabilizer Codes and Algebraic Curves

Let us now specialize a bit further. Our aim will be to introduce quantum stabilizer codes and how to construct them using algebraic curves over finite fields. For disussion of quantum stabilizer codes, see reference [320].

The stabilizer of a given a subgroup of $S \subset \mathcal{G}_n$, provides a possible quantum analog to linear codes. We define a q-ary quantum stabilizer code C of length n as the joint eigenspace of operators, i.e.:

$$Q = \text{Stab}(S) = \left\{ |u\rangle \in (\mathbb{C}^q)^{\otimes n} \quad \text{with} \quad M|u\rangle = |u\rangle \quad \text{for all } M \in S \right\}. \tag{F.28}$$

The interesting point for us is that algebraic curves over finite fields provide a natural way to specify subgroups of $S \subset \mathcal{G}_n$, and consequently, give us a way to build quantum stabilizer codes. This is often referred to as the CSS construction, after references [113, 114].

To begin, introduce a quadratic field extension $\mathbb{F}_{q^2} = \mathbb{F}_q(\omega)$ over $\mathbb{F}_q$. We can take as basis vectors ω and $\omega^q \equiv \overline{\omega}$ and present elements of $\mathbb{F}_{q^2}^n$ as linear combinations $\omega a + \overline{\omega} b$ for $a, b \in \mathbb{F}_q^n$. Given this, we can perform the following composition of maps:

$$\mathbb{F}_{q^2}^n \xrightarrow{f} \mathbb{F}_q^{2n} \xrightarrow{g} \mathcal{G}_n \tag{F.29}$$

$$\omega a + \overline{\omega} b \mapsto (a; b) \mapsto E_{ab}. \tag{F.30}$$

So, given a suitable linear code subspace $C \subset \mathbb{F}_q^{2n}$, the image space $g(C)$ gives a collection of error correction operations. The stabilizer $Q = \text{Stab}(g(C))$ then defines a quantum stabilizer code.

To make use of this in explicit constructions, we can introduce a notion of an orthogonal dual, as specified by making a choice of bilinear pairing. Two canonical options are a symplectic pairing and a Hermitian pairing. For the symplectic pairing, we use $\mathbb{F}_{q^2}^n \simeq \mathbb{F}_q^{2n}$ and assume $u = \omega a + \overline{\omega} b$ and $v = \omega a' + \overline{\omega} b'$. Then, we can specify the product as:

$$(\omega a + \overline{\omega} b) *_s (\omega a' + \overline{\omega} b') = \mathrm{Tr}\left(\sum_{i=1}^{n} a_i b_i' - b_i a_i'\right). \tag{F.31}$$

Given $u, v \in \mathbb{F}_{q^2}^n$, we denote the Hermitian pairing as the $\mathbb{F}_q$ valued expression:

$$u *_h v = \sum_{i=1}^{n} u_i v_i^q = \sum_{i=1}^{n} u_i \overline{v_i}. \tag{F.32}$$

Observe that orthogonality in the Hermitian pairing implies orthogonality with respect to the symplectic pairing.

Given a vector space $C \subset \mathbb{F}_q^{2n}$, we can then consider, for each choice of pairing, the space of vectors which are "orthogonal" to C with respect to the symplectic pairing as:

$$C^{(s)} = \left\{ v \in \mathbb{F}_q^{2n} \quad \text{such that} \quad v *_s c = 0 \text{ for all } c \in C \right\} \tag{F.33}$$

Given a vector space $C \subset \mathbb{F}_{q^2}^n$, we can reference the space of vectors which are "orthogonal" to C with respect to the Hermitian pairing as:

$$C^{(h)} = \left\{ v \in \mathbb{F}_{q^2}^n \quad \text{such that} \quad v *_h c = 0 \text{ for all } c \in C \right\}. \tag{F.34}$$

An important general result due to reference [115] is that with $q = p^m$ and for $C \subset \mathbb{F}_q^{2n}$ an $\mathbb{F}_p$-linear code of order p^r which is self-orthogonal with respect to the $*_s$ product, namely $C \subset C^{(s)}$, then any eigenspace of the CSS map $g(C)$ is a $[[n, n - \frac{r}{m}, d(C^{(s)} \backslash C)]]_q$ code. Note that we implicitly have r/m an integer, and in many applications one makes the further assumption that C is an $\mathbb{F}_q$-linear code, defining a vector space of dimension $k = r/m$.

There are various immediate corollaries of this result. For example, we can now specify two linear codes C_1 and C_2 of length n and respective dimensions k_1 and k_2 with $C_1 \subset C_2$. Then, using the standard dot product of line (F.14), we construct the dual space $C_2^\perp$. We can then produce a subspace $C = \omega C_1 + \overline{\omega} C_2^\perp \subset \mathbb{F}_{q^2}^n$. Observe that $f(C) = D$ is self-orthogonal, with $f : \mathbb{F}_{q^2}^n \to \mathbb{F}_q^{2n}$ defined in line (F.29). This can be used to prove that the eigenspace of $g(D)$ is in fact a quantum stabilizer code, and implicitly specifies an $[[n, k, d]]_q$ code with $k = k_2 - k_1$ and $d = \min\{d(C_2 \backslash C_1), d(C_1^\perp \backslash C_2)\}$.

We can obtain a similar set of assertions using the Hermitian pairing and a classical q^2-ary linear code $[n, k, d]_{q^2}$, where we assume that the associated vector space C is self-orthogonal with respect to the Hermitian pairing. Denoting this orthogonal space by $C^{(h)}$, the resulting quantum code constructed from this data is an $[n, n - 2k, \min\{\mathrm{wt}(C^{(h)} \backslash C)\}]]_q$ code.

Having seen how to build quantum stabilizer codes from classical codes, we next observe that classical algebraic codes provide a way to generate many examples of such quantum stabilizer codes. Additionally, some of the conditions implcitly used, such as the condition that we find two linear codes C_1 and C_2 such that $C_1 \subset C_2$ simply amount to specifying line bundles in the appropriate fashion. For example, using the fact that for divisors $A \leq B$ (namely $B - A$ is effective) we observe that the line bundles satisfy $\mathcal{L}(A) \subset \mathcal{L}(B)$, and so we also have $C_{\mathcal{L}}(D, A) \subset C_{\mathcal{L}}(D, B)$ for the corresponding linear codes, where $D = P_1 + ... + P_n$ is the divisor given by our "evaluation points".

Summarizing, we have discussed a few ways to generate quantum stabilizer codes. In fact, the interesting feature of these examples is that we also establish the existence of families of quantum codes, since we already have families of classical linear codes.

G Partition Function for a Free Field

In this Appendix we study the partition function for a free field defined on the punctured affine line $\mathbb{A}^\times$. Our Lagrangian is given by:

$$L[\phi] = \kappa D_u \phi D_u \phi, \tag{G.1}$$

and our task will be to evaluate the corresponding path integral:

$$Z\left[\mathbb{A}^\times\right] \equiv \sum_\phi \exp\left(\frac{2\pi i}{p} S[\phi]\right). \tag{G.2}$$

We comment here that when we turn to fermionic and supersymmetric systems, we introduce a separate notion of a partition function as associated with the Hasse-Weil Zeta function. The interpretation of the two is somewhat different, but the context should also make clear quantity we are considering.

Now, to carry out the evaluation of the path integral, we make use of the mode expansion:

$$\phi(u) = \sum_m \phi_m u^m \in \mathbb{F}_p[u, u^{-1}] \subset \mathbb{F}_p[[u, u^{-1}]], \tag{G.3}$$

and we observe that the u^m are eigenfunctions of the differential operator D_u^2 with eigenvenalues m^2. So, we get the momentum space expression:

$$S[\phi] = \sum_{m,n} -\kappa m n \widehat{\delta}_{m+n} \phi_m \phi_n, \tag{G.4}$$

where $\widehat{\delta}_{m+n}$ is the modified delta function which only enforces $m + n = 0 \bmod (p-1)$. Compared with the characteristic zero answer, we see the same subtlety encountered in our evaluation of correlation functions: we have to contend with the "winding modes" present in the characteristic p setting since the delta function only enforces a milder version of momentum conservation.

Proceeding much as in section 8, we introduce (see equation (8.18)):

$$\phi_m^\alpha \text{ modes: } m \in \{1, ..., p-1\} \quad \text{and} \quad \alpha \in \mathbb{Z}, \tag{G.5}$$

as well as vectors μ_m^α with:

$$\mu_m^\alpha = (m - \alpha), \tag{G.6}$$

which we write as $\overrightarrow{\mu}_m$, namely a vector in the α index. Then, the action is given by:

$$S[\phi] = \sum_{m=1}^{p-1} -\kappa \left(\overrightarrow{\mu}_m \cdot \overrightarrow{\phi}_m\right) (\overrightarrow{\mu}_{-m} \cdot \overrightarrow{\phi}_{-m}), \tag{G.7}$$

in the obvious notation. Here, we permit the same abuse of notation to allow the index "$-m$" to go out of range.

Recall that in our analysis of correlation functions we first isolated the contributions from the zero modes. Here, we must perform a similar analysis, and consequently need to also split up our analysis according to whether $\vec{\mu}_m \cdot \vec{\mu}_m \neq 0$ or $\vec{\mu}_m \cdot \vec{\mu}_m = 0$. Returning to our discussion in section 8, we have the expansion:

$$\vec{\phi}_m = a_m \vec{\nu}^{(0)}_m + \sum_{l \neq 0} a^{(l)}_m \vec{\nu}^{(l)}_m, \tag{G.8}$$

in which:

$$\vec{\mu}_m \cdot \vec{\phi}_m = a_m. \tag{G.9}$$

The range of values for a_m are the p distinct values in the finite field $\mathbb{F}_p$. These are the propagating degrees of freedom in the model. Moreover, when $\vec{\mu}_m \cdot \vec{\mu}_m \neq 0$ we can also set $\vec{\nu}^{(0)}_m = \vec{\mu}_m$. Our expression for the action is therefore given by:

$$S[\phi] = \sum_{m=1}^{p-1} -\kappa a_m a_{-m} \left(\vec{\nu}^{(0)}_m \cdot \vec{\nu}^{(0)}_m\right) \left(\vec{\nu}^{(0)}_{-m} \cdot \vec{\nu}^{(0)}_{-m}\right). \tag{G.10}$$

To proceed further, observe that in our path integral, we wish to sum over all possible values of a_m. Consider fixing a value of a_m. Then, there exists a unique $b_m \in \mathbb{F}_p$ such that $b_m = a_m(\vec{\nu}^{(0)}_m \cdot \vec{\nu}^{(0)}_m)$, and so instead of summing over a_m, it is enough to sum over b_m. With this in place, we can now proceed to the evaluation of the path integral sum. Precisely because the path integral splits up as a sum over the different $\vec{\phi}_m$ modes, we get:

$$Z\left[\mathbb{A}^{\times}\right] = \sum_{\vec{\phi}_m} \exp\left(\frac{2\pi i}{p} \sum_{m=1}^{p-1} -\kappa \left(\vec{\mu}_m \cdot \vec{\phi}_m\right) \left(\vec{\mu}_{-m} \cdot \vec{\phi}_{-m}\right)\right) \tag{G.11}$$

$$= |\mathbb{F}_p|^{\sigma} \times \sum_{a_1 \in \mathbb{F}_p} ... \sum_{a_{p-1} \in \mathbb{F}_p} \exp\left(\frac{2\pi i}{p}\left(\sum_{m=1}^{p-2} -\kappa a_m a_{p-1-m}\right) - \frac{2\pi i}{p}\kappa a^2_{p-1}\right), \tag{G.12}$$

where the prefactor $|\mathbb{F}_p|^{\sigma}$ is the contribution from the zero modes. This is formally infinite, but serves to remind us that most of the path integral sums are trivial. Here, we have also used the identity $a_{p-1-m} = a_{-m}$ for $m = 1, ..., p-2$ and $a_{-(p-1)} = a_{p-1}$.

We can also further simplify the sum over $a_m a_{p-m}$. Indeed, we have:

$$\sum_{m=1}^{p-1} a_m a_{p-m} = a^2_{p-1} + \sum_{m=1}^{p-2}\left(\left(\frac{a_m + a_{p-m}}{2}\right)^2 - \left(\frac{a_m - a_{p-m}}{2}\right)^2\right). \tag{G.13}$$

We can re-index these variables to produce a new basis to sum over:

$$c_k = \frac{a_k + a_{p-k}}{2} \quad \text{for} \quad k = 1, ..., \frac{p-1}{2} \tag{G.14}$$

$$d_k = \frac{a_k - a_{p-k}}{2} \quad \text{for} \quad k = 1, ..., \frac{p-1}{2}, \tag{G.15}$$

so that our path integral sum becomes:

$$Z\left[\mathbb{A}^\times\right] = |\mathbb{F}_p|^\sigma \times \sum_{a_{p-1}\in\mathbb{F}_p} \sum_{c_1\in\mathbb{F}_p} ... \sum_{c_{\frac{p-1}{2}}\in\mathbb{F}_p} \sum_{d_1\in\mathbb{F}_p} ... \sum_{d_{\frac{p-1}{2}}\in\mathbb{F}_p} \exp\left(\frac{2\pi i}{p} - \kappa a_{p-1}^2 \sum_{k=1}^{(p-1)/2} - \kappa\left(c_k^2 - d_k^2\right)\right). \tag{G.16}$$

At this point we recognize that each sum over a a_{p-1}, c_k and d_k is producing a standard quadratic Gaussian sum:

$$g(\kappa; p) = \sum_{n=0}^{p-1} \exp\left(\frac{2\pi i}{p}\kappa n^2\right). \tag{G.17}$$

We have the well-known relation for κ not divisible by $p \neq 2$:

$$g(\kappa; p) = \left(\frac{\kappa}{p}\right) g(1; p), \tag{G.18}$$

where $\left(\frac{\kappa}{p}\right)$ is the Legendre symbol.[109] Moreover, we have:

$$g(1; p) = \left\{ \begin{array}{ccc} \sqrt{p} & \text{if} & p = 1 \bmod 4 \\ -i\sqrt{p} & \text{if} & p = 3 \bmod 4 \end{array} \right\}. \tag{G.19}$$

Putting everything together, we conclude that our partition function is simply a formal product over quadratic Gaussian sums. Indeed, returning to equation (G.16) we can now write:

$$Z\left[\mathbb{A}^\times\right] = |\mathbb{F}_p|^\sigma \times g(-\kappa; p) \times \left(g(\kappa; p)g(-\kappa; p)\right)^{(p-1)2}, \tag{G.20}$$

or:

$$Z\left[\mathbb{A}^\times\right] = |\mathbb{F}_p|^\sigma \times \left(\frac{1}{p}\right)^{(p-1)/2} \left(\frac{-1}{p}\right)^{(p+1)/2} \left(g(1; p)\right)^p. \tag{G.21}$$

Let us comment that generalizing to the contribution with a background source term follows in a similar manner. Indeed, returning to our discussion in section 8, we simply need

[109] Recall that the Legendre symbol $\left(\frac{\kappa}{p}\right)$ is, for p an odd prime and κ non-zero mod p, given by 1 if κ is a quadratic residue mod p and -1 if κ is not a quadratic residue mod p. The Legendre symbol $\left(\frac{\kappa}{p}\right)$ is zero if κ vanishes mod p. Here, "quadratic residue" simply means there exists an x such that $x^2 = \kappa$ mod p.

to add in the contribution from the vev of $\mathcal{O}_J$:

$$\mathcal{O}_J = \exp\left(\frac{2\pi i}{p}\sum_{m=1}^{p-1}\overrightarrow{J}_m \cdot \overrightarrow{\phi}_m\right). \tag{G.22}$$

In the obvious notation, we have:

$$Z\left[\mathbb{A}^{\times}, J\right] = Z\left[\mathbb{A}^{\times}\right] \times \langle \mathcal{O}_J \rangle\,, \tag{G.23}$$

where, $\langle \mathcal{O}_J \rangle$ is given in equations (8.41) and (8.42)).

H Alternative Action on $\mathbb{A}^{\times}$

In the main body of this paper we have presented a proposal for constructing and evaluating actions. The main idea in this approach is to view the field configurations as rational morphisms $\phi : X \dashrightarrow Y$ between characteristic p varieties, and to use an explicit evaluation map to sum over the geometric points of the "support spacetime X". For example, the action of a free Gaussian field on the punctured affine line $\mathbb{A}^{\times}$ can be written as:

$$S[\phi] = \sum_{x \in \mathbb{A}^{\times}} \mathrm{ev}_{u=x} \kappa D_u \phi D_u \phi, \tag{H.1}$$

for $\phi \in \mathbb{F}_p[u, u^{-1}]$ and $D_u = u\partial_u$. Now, in our discussion of section 8 we also saw that there is also a mode expansion available which parallels the mode expansions encountered in characteristic zero. One of the distinct features of working in characteristic p, however, is that the resulting action only led to a milder form of orthogonality. Indeed, evaluating on a finite point set, we found that there are many additional contributions since $x^{m+p} = x^m$. It is therefore natural to ask whether one could "do better" by using a different technique for constructing actions.

At least in specialized situations such as $\mathbb{A}^{\times}$, there is indeed a natural generalization availalable which hews more closely to the characteristic zero treatment. The reason we have not adopted this perspective throughout is that it is not clear to us that it naturally extends to geometries such as the affine line, or more general genus g curves. We return to this point after first providing additional details on this "alternative treatment".

The alternative method for proceeding is to simply introduce a map on $\mathcal{O}_{\mathbb{A}^{\times}} = \mathbb{F}_p[u, u^{-1}]$ given by evaluation on the degree zero terms:

$$\eta : \mathcal{O}_{\mathbb{A}^{\times}} \to \mathbb{F}_p \tag{H.2}$$

$$\sum \phi_m u^m \mapsto \phi_0. \tag{H.3}$$

Observe that in evaluating on the kinetic term $D_u\phi D_u\phi$ as well as various interaction terms, this returns:

$$\eta \left(D_u \phi D_u \phi \right) = \sum_m - m^2 \phi_{-m} \phi_m \tag{H.4}$$

$$\eta \left(\phi^4 \right) = \sum_{m+n+r+s=0} \phi_m \phi_n \phi_r \phi_s. \tag{H.5}$$

All of this closely parallels the characteristic zero case, and the notion of conservation of momentum also carries over as well.

Why then, have we not adopted this treatment in the main body of this paper? There are at least a few difficulties which make it difficult to generalize this construction to other settings. Let us now enumerate some of these difficulties.

First, there is the issue of how to define the analog of $\eta : \mathcal{O}_{\mathbb{A}^\times} \to \mathbb{F}_p$ for more general varieties X. One can already see the issue in comparing the coordinate ring for the affine line $\mathcal{O}_{\mathbb{A}^1} = \mathbb{F}_p[u]$, with that of $\mathcal{O}_{\mathbb{A}^\times} = \mathbb{F}_p[u, u^{-1}]$. On $\mathbb{F}_p[u]$, restricting to degree zero terms is extremely restrictive, and would not lead to a particularly natural notion of a kinetic term or interaction term. One might attempt to instead consider a canonical pairing (via a suitable application of Serre duality) to construct a pairing $\mathcal{O}_{\mathbb{A}^1} \times \mathcal{O}_{\mathbb{A}^1} \to \mathbb{F}_p$ in which only terms of the same degree are kept. But this is also a bit awkward, especially when one turns to general interaction terms, where a pairing is not always available.

Second, there is the question of whether the characteristic zero intuition should just be carried over completely from the start. We have remarked many times that one of the intriguing features of characteristic p geometry is the fact that $x^p = x$ for $x \in \mathbb{F}_p$. In particular, we have argued that this implicitly imposes important truncations on physically distinct configurations, serving as a generalized cutoff on the UV degrees of freedom in our system. Attempting to implement a map such as $\eta : \mathcal{O}_{\mathbb{A}^\times} \to \mathbb{F}_p$ goes counter to this philosopy, since it treats each mode number as wholly independent, rather than correlated.

Finally, there is the issue as to how we should generalize this to other geometries. At least when we evaluate on geometric points of X, there is a notion of explicitly returning an element of $\mathbb{F}_p$. In the broader setting where we need to match up explicit "Fourier modes", one faces the related question as to whether one has secretly made use of a preferred basis of functions. This is not much of an issue in "simple" geometries, but it is worth remembering that at least in characteristic zero, the choice of preferred mode expansion basis can be quite sensitive to the local geometry.

All that being said, it would of course be interesting to study further actions constructed in this way, though as we have already mentioned, it is unclear to us that it fruitfully generalizes.

I Inverse Limits

In this Appendix we briefly review the notion of an "inverse limit". This material can be found in standard abstract algebra textbooks, including for example [307, 321]. As we are mathematical dilettantes, we will content ourselves to closely follow reference [322].

We begin with a collection of groups A_i indexed by $i \in I$ such that there is a directed ordering $\leq$ on the partially ordered set I. For our purposes, we can typically take this to just be the natural numbers $\mathbb{N}$. For $i \leq j$, introducing "bonding maps" given by group homomorphisms:

$$f_{ij} : A_j \to A_i \tag{I.1}$$

where we demand that f_{ii} is just the identity and $f_{ij} \circ f_{jk} = f_{ik}$ with $i \leq j \leq k$. This collection of data defines an inverse system. This readily generalizes to other algebraic / topological structures.

To construct an inverse limit, we consider the direct product over all the A_i as well as sequences:

$$\overrightarrow{a} \in \prod_{i \in I} A_i. \tag{I.2}$$

We denote by a_i the component of the vector in A_i. The inverse limit for this system is then specified by the condition that these components of the vector are compatible with the bonding maps, i.e., we have $a_i = f_{ij}(a_j)$. The resulting set of sequences are the elements of the inverse limit:

$$\varprojlim_{i \in I} A_i \equiv \left\{ \overrightarrow{a} \in \prod_{i \in I} A_i \;\; \text{such that} \;\; a_i = f_{ij}(a_j) \right\}. \tag{I.3}$$

The same sort of construction holds for more general sorts of maps, including ring and field homomorphisms. Since it is often clear from the context, we often leave the indexing set implicit, as we have done in the main body of the text.

An important example of an inverse limit includes the construction of $\mathbb{Z}_p$, the p-adic ring of integers from the rings $\mathbb{Z}/p^n\mathbb{Z}$. In this construction, one takes a sequence of integers $(m_1, m_2 ...,) = \{m_i\}_{i \in I}$ such that $m_i = m_j \mod p^i$ for $i < j$. This can also be equipped with the product topology, with open sets given by cylinder sets.[110]

Let us also remark that the inverse limit also extends categorically, i.e., we have $\varprojlim^n = R^n \varprojlim$, where R^n denotes the nth right-derived functor on a category with a notion of $\varprojlim$.

[110] The cylinder sets are defined as follows. Begin with a Cartesian product as indexed by some set S:

$$X = \prod_{s \in S} Y_s \tag{I.4}$$

and consider the projections:

$$\pi_s : X \to Y_s \tag{I.5}$$

. A cylinder set in X consists of intersections of the pre-images of these projections.

J The Étale Fundamental Group

In this Appendix we briefly review some aspects of the étale fundamental group of a scheme X. Some of what we describe is reviewed in [323] as well as the notes [324]. For a generalization to higher homotopy groups, see, e.g., Appendix E of reference [128].

We first introduce a scheme X and fix a geometric point x, which in the topological case we would view as the base point for our loops. The definition of $\pi_1^{\text{ét}}(X, x)$ follows from taking an inverse limit of Galois covers $X_i \to X$, i.e., we first consider finite étale schemes $X_i \to X$, along with a projective system $\{X_i \to X_j | i < j \in I\}$ for some ordered indexing set I. Then, we build the étale fundamental group as the inverse limit:

$$\pi_1^{\text{ét}}(X, x) = \varprojlim_{i \in I} \mathrm{Aut}_X(X_i). \tag{J.1}$$

One can visualize this as the group of deck transformations on a covering space. The reason we need to introduce an inverse limit is that the notion of a "universal cover" may not be available in the more general setting. What this means in practice is that when the standard fundamental group of a topological space exists, the étale fundamental group is a profinite completion of the standard fundamental group.[111]

To illustrate the general idea, consider the case where the scheme X is just $\mathrm{Spec}\, K$, with K a field. Geometrically, this describes a single point (since a field has a single geometric point), so even though "topologically" we might view this as having a trivial fundamental group, its loop space is still quite non-trivial. Indeed, in this case, our Galois covers really are just Galois extensions of K, and the inverse limit is obtained from a separable algebraic closure of K, which we denote by $\overline{K}$. Then, we can write:

$$\pi_1^{\text{ét}}(\mathrm{Spec}\, K, x) = \mathrm{Gal}(\overline{K}/K), \tag{J.3}$$

namely the absolute Galois group of K. To be more concrete, we can also specialize to the case of $K = \mathbb{F}_q$. Then, the étale fundamental group is:

$$\pi_1^{\text{ét}}(\mathrm{Spec}\, \mathbb{F}_q, x) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}}, \tag{J.4}$$

namely the profinite completion of the integers, i.e., the profinite completion of the funda-

[111] Recall that a profinite completion of a group G is specified by first considering the inverse system built from all quotients G/N with N a normal subgroup of G of finite index. These quotients form an inverse system because the normal subgroups form a partial ordering under inclusion. Then, the profinite completion is defined by:

$$\widehat{G} \equiv \varprojlim G/N. \tag{J.2}$$

For example, $\widehat{\mathbb{Z}}$, the group of profinite integers, is obtained from the inverse limit on $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$, where the partial ordering (and thus inclusion) proceeds via the embedding $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m|n$. We comment that $\widehat{\mathbb{Z}}$ is the absolute Galois group for any finite field.

mental group of a topological circle, namely an S^1. The generator of this group is just the q^{th} Frobenius map $t \mapsto t^q$. Continuing the analogy further, once can view $\mathbb{F}_p$ as specifying a circle in $\operatorname{Spec}\mathbb{Z}$, which is then interpreted as an arithmetic analog of a topological three-sphere, namely an S^3 (see e.g. [325, 91]).

Similar considerations hold for the projective line $\mathbb{P}^1(K)$, since in this case the functions of a local system are captured by rational functions of polynomials. In this case, we have

$$\pi_1^{\text{ét}}(\mathbb{P}^1(K), x) = \mathrm{Gal}(\overline{K(t)}/K(t)) \simeq \mathrm{Gal}(\overline{K}/K). \tag{J.5}$$

In particular, in the case where we specialize to $K = \mathbb{F}_q$, we get:

$$\pi_1^{\text{ét}}(\mathbb{P}^1(\mathbb{F}_q), x) \simeq \widehat{\mathbb{Z}}, \tag{J.6}$$

so it is sensible to view this as specifying a notion of "winding numbers" in the characteristic p setting.

In the case of the affine line $\mathbb{A}^1(K)$, the computation of the étale fundamental group is somewhat more complicated because there are now far more non-trivial étale coverings available. For a recent account of some of the issues involved, see for example reference [326].

K Some Zeta Functions

In this Appendix we collect a few examples of Zeta functions. To begin, we fix our ground field to be $\mathbb{F}_q$, and assume (as usual) that q is odd for simplicity. As a first example, consider the variety $\mathbb{A}^n$, i.e., affine n-dimensional space. Counting points in this setting is straightforward, and we get:

$$Z_{\mathbb{A}^n,q}(z) = \frac{1}{1-q^n z}. \tag{K.1}$$

The case of projective n-dimensional space is similar, and gives:

$$Z_{\mathbb{P}^n,q}(z) = \frac{1}{(1-z)(1-qz)...(1-q^n z)}. \tag{K.2}$$

As a somewhat more involved example discussed in reference [151], we next consider the elliptic curve $\mathbb{E}$ defined as a zero set in $\mathbb{P}^2$:[112]

$$Y^2 Z = X^3 + XZ^2, \tag{K.3}$$

in the obvious notation. The Zeta function in this case is:

$$Z_{\mathbb{E},q}(z) = \frac{1 - az + qz^2}{(1-z)(1-qz)} = (1 - az + qz^2) Z_{\mathbb{P}^1,q}(z), \tag{K.4}$$

where the number a is implicitly fixed by the relation:

$$\#(\mathbb{E}) = -a + 1 + q, \tag{K.5}$$

where $\#(\mathbb{E})$ is the number of points in $\mathbb{E}$ defined over $\mathbb{F}_q$. Note also that the denominator is the same as that of $Z_{\mathbb{P}^1,q}(z)$. An additional remark here is that for this curve, the rigid cohomology group is [151, 152]:

$$H^1_{\text{rig}}(\mathbb{E}) \simeq \mathbb{Q}_q \frac{dx}{y} \oplus \mathbb{Q}_q x \frac{dx}{y}, \tag{K.6}$$

where $\mathbb{Q}_q$ with $q = p^n$ denotes the degree n unramified extension over the p-adics $\mathbb{Q}_p$, i.e., we have $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \simeq \text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \simeq \mathbb{Z}/n\mathbb{Z}$.[113]

One can also consider the affine case, i.e., by setting $Z = 1$ in equation (K.3) and excluding $y = 0$. This yields [151]:

$$Z_{\mathbb{E}_{\text{aff}},q}(z) = Z_{\mathbb{E},q}(z)(1-z)^2(1-z^2), \qquad \text{if } q \equiv -1 \bmod 4 \tag{K.7}$$

$$Z_{\mathbb{E}_{\text{aff}},q}(z) = Z_{\mathbb{E},q}(z)(1-z)^4, \qquad \text{if } q \equiv +1 \bmod 4. \tag{K.8}$$

[112] There is an unfortunate clash of notation between the projective coordinate and the Zeta function. It should be clear from the context which is meant.

[113] See Appendix O for a brief review of $\mathbb{Q}_p$ and $\mathbb{Q}_q$.

L Real and Complex Spacetime Twistors

In this Appendix we review some aspects of the geometry of real and complex twistors [327, 328], and their use in the study of Lorentzian signature spacetimes. Twistor methods are helpful in addressing a number of Euclidean signature issues, for example, in generating self-dual solutions to Yang-Mills theory [329].[114] Our emphasis here will be on the usage of physical twistors in characterizing the conformal structure of spacetime. By recasting this as a problem in algebraic geometry, we can then consider varying the ground field.[115]

The main idea in this approach is to emphasize the conformal structure of a spacetime. We denote by $\mathbb{RM}$ four-dimensional Minkowski space, and $\mathbb{CM}$ its complexification. The conformal compactification of each space is denoted by $\mathbb{RM}^{\#}$ and $\mathbb{CM}^{\#}$. The space $\mathbb{CM}^{\#}$ is characterized by a quadric in $\mathbb{CP}^5$. To see how this comes about, introduce six independent homogeneous coordinates for $\mathbb{CP}^5$ which we label as $R^{\alpha\beta}$ where $\alpha, \beta = 1, ..., 4$ and $R^{\alpha\beta} = -R^{\beta\alpha}$. Raising of the indices is accomplished via the ε tensor:

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta} = R_{\gamma\delta}. \tag{L.1}$$

The quadric corresponding to $\mathbb{CM}^{\#}$ is then:

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}R^{\gamma\delta} = R_{\gamma\delta}R^{\gamma\delta} = 0. \tag{L.2}$$

The important point for us is that this discussion makes no reference to an explitic metric. A real slice of this quadric defines $\mathbb{RM}^{\#}$ via the coordinate substitution:[116]

$$R^{12} = \frac{1}{2}(V+W),\ R^{13} = \frac{1}{\sqrt{2}}(Y-iX),\ R^{14} = \frac{i}{\sqrt{2}}(T+Z) \tag{L.3}$$

$$R^{23} = \frac{i}{\sqrt{2}}(Z-T),\ R^{24} = \frac{1}{\sqrt{2}}(Y+iX),\ R^{34} = V-W \tag{L.4}$$

where the variables T, V, W, X, Y and Z are real coordinates of $\mathbb{R}^{2,4}$. In terms of these

[114] A curiosity of twistors is that physics in Kleinian signature $(+,+,-,-)$ is "better-behaved" than one might initially suspect. For example, many analytic aspects of scattering amplitudes and supersymmetric structures persist and have novel properties in this setting, see e.g., references [138, 330, 137, 331, 332] and references therein for discussion on various aspects of these applications.

[115] The following references provide some additional context and details [134–136, 333]. Some notions of p-adic twistor space have been discussed for example in reference [334]. Here, we are considering what happens to physical twistors when varying the ground field. The point of view in [334] is more in line with its application to questions in Euclidean signature systems where one is often interested in tracking the local variation of families of complex structures. There is then an analogous variational problem one can formulate p-adically. Our interest and point of view is somewhat different.

[116] In the context of working over a finite field, the appearance of $\sqrt{2}$ and i is potentially in poor taste. We can, of course, absorb these factors into the definitions of the coordinates. Observe that in $\overline{\mathbb{F}}_p$ there is no such issue, though the interpretation is of course somewhat different in characteristic p.

coordinates, equation (L.2) becomes:

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0. \tag{L.5}$$

The spacetime $\mathbb{RM}^{\#}$ has topology $S^1 \times S^3$, which is the same as the Euclidean signature Einstein Universe.

Deleting an appropriate subspace from $\mathbb{RM}^{\#}$ yields de Sitter space (dS), Anti de Sitter space (AdS) and Minkowski space via the restrictions:[117]

$$\text{dS: } T = l \tag{L.6}$$
$$\text{AdS: } W = l \tag{L.7}$$
$$\text{Minkowski: } V - W = 0. \tag{L.8}$$

In the complexified setting, the subspace to be deleted from $\mathbb{CM}^{\#}$ is:

$$I_{\alpha\beta} R^{\alpha\beta} = 0 \tag{L.9}$$

where the bitwistor $I_{\alpha\beta}$ is sometimes referred to as the "infinity twistor". For the three spacetimes in question, the tensor $I_{\alpha\beta}$ satisfies:

$$\text{dS: } I_{\alpha\beta} I^{\alpha\beta} = \frac{2}{l^2} \tag{L.10}$$
$$\text{AdS: } I_{\alpha\beta} I^{\alpha\beta} = -\frac{2}{l^2} \tag{L.11}$$
$$\text{Mink: } I_{\alpha\beta} I^{\alpha\beta} = 0. \tag{L.12}$$

Explicit representatives for each spacetime are:

$$I_{\alpha\beta}^{(dS)} = \frac{1}{l}\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{L.13}$$

$$I_{\alpha\beta}^{(AdS)} = \frac{1}{l} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & +\frac{1}{2} & 0 \end{bmatrix} \tag{L.14}$$

[117] Observe that these restrictions make manifest the $SO(1,4)$, $SO(2,3)$ and $SO(1,3)$ symmetries of these spacetimes since the defining equations are invariant under these group actions.

$$I^{(M)}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{L.15}$$

One of the advantages of working with respect to the conformal compactification is that it makes manifest the asymptotic behavior of the spacetime in question. In subsequent sections we shall further review this behavior.

We now turn to the characterization of these spacetimes in terms of twistor space. First, observe that the quadric equation is automatically satisfied by making the substitution:

$$R^{\alpha\beta} = Z^{\alpha}W^{\beta} - Z^{\beta}W^{\alpha} \tag{L.16}$$

where Z^{α} and W^{α} are homogeneous coordinates of two points in (complex) projective twistor space $\mathbb{PT}^{\bullet}$, which we view as a copy of $\mathbb{CP}^3$. Said differently, a pair of points in twistor space yields a single point in spacetime.

Conversely given a point in $\mathbb{CM}^{\#}$, this defines a $\mathbb{CP}^1$ in $\mathbb{PT}^{\bullet}$ via the incidence relation:

$$R^{[\alpha\beta}Z^{\gamma]} = 0 \tag{L.17}$$

which is equivalent to the condition:

$$R_{\alpha\beta}Z^{\beta} = 0. \tag{L.18}$$

Written out as a matrix, we have:

$$\begin{bmatrix} 0 & R^{34} & -R^{24} & R^{23} \\ -R^{34} & 0 & R^{14} & -R^{13} \\ R^{24} & -R^{14} & 0 & R^{12} \\ -R^{23} & R^{13} & -R^{12} & 0 \end{bmatrix} \begin{bmatrix} Z^1 \\ Z^2 \\ Z^3 \\ Z^4 \end{bmatrix} = 0. \tag{L.19}$$

Although this is four equations, only two are actually independent. Indeed, treating $R_{\alpha\beta}$ as a 4×4 matrix, we observe that the determinant is:

$$\det R_{\alpha\beta} = \left(\varepsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta}R_{\gamma\delta}\right)^2 = 0. \tag{L.20}$$

Since $R_{\alpha\beta}$ is anti-symmetric, we conclude that $R_{\alpha\beta}$ is a rank two matrix. In other words, the incidence relation defines two divisor equations inside of $\mathbb{CP}^3 = \mathbb{PT}^{\bullet}$, and their intersection defines a $\mathbb{CP}^1$, specified by $R^{\alpha\beta}$.

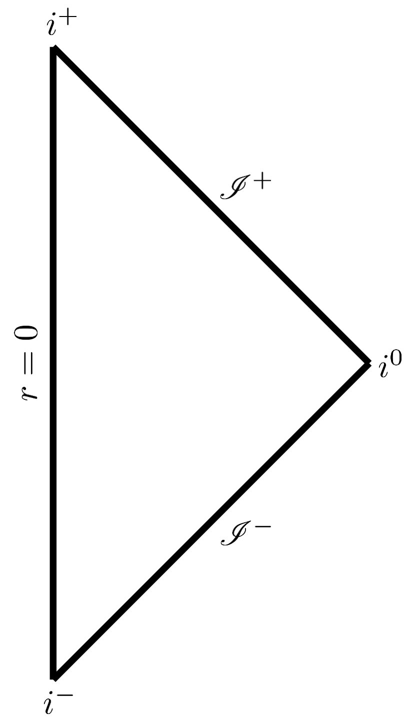

Figure 14: Penrose diagram for 4D Minkowski space, with metric $ds^2 = dt^2 - dr^2 - r^2 d\Omega_2^2$, where $-\infty < t < \infty$ and $0 \leq r < \infty$ and $d\Omega_2^2$ denotes the metric for a unit radius S^2 (we hope our choice of sign conventions for the metric is not too distracting). Each point in the interior represents an S^2. We have spatial infinity at i^0, timelike future infinity at i^+, timelike past infinity at i^-. Future null infinity is at $\mathscr{I}^+$ and past null infinity is at $\mathscr{I}^-$. There is only a coordinate singularity at $r = 0$, so one can extend to a bigger region as specified by a square.

L.1 Minkowski Space

Let us now turn to a further discussion of Minkowski space. Topologically, this spacetime is given by $\mathbb{R} \times \mathbb{R}^3$. There are a few special regions "at infinity", as captured by the Penrose diagram. We have timelike future infinity i^+ and timelike past infinity i^-. We also have spatial infinity i^0. The segment joining i^0 to i^+ (resp. i^-) defines future (resp. past) null infinity $\mathscr{I}^+$ (resp. $\mathscr{I}^-$). Future null infinity has the topology of $\mathbb{R} \times S^2$. In $\mathbb{RM}^{\#}$, i^0, i^+, i^- are all identified with a single point, which is referred to as the "point at infinity" which we denote by i^∞. See figure 14 for a depiction of the Penrose diagram.

All of the points at infinity are characterized by the hyperplane $V - W = 0$. The point i^∞ is distinguished as the unique point of $V - W = 0$ which has tangent space equal to the quadric of equation (L.5). Indeed, the unique differential element for the two defining equations is:

$$dF_{quadric} = 2\left(TdT + VdV - WdW - XdX - YdY - ZdZ\right) \tag{L.21}$$

$$dF_{V-W} = dV - dW \tag{L.22}$$

which are parallel at the point $T = X = Y = Z = (V - W) = 0$. The points of $\mathscr{I}^+$ and $\mathscr{I}^-$ correspond to those points on the intersection of $V - W$ and the quadric which are distinct from this point of tangency.

Let us now turn to a twistor characterization of the points at infinity for Minkowski space. The locus of points to be deleted is

$$\text{Delete: } I^{(M)}_{\alpha\beta} R^{\alpha\beta} = 2R^{34} = 0. \tag{L.23}$$

Along this locus, the quadric equation reduces to:

$$R^{34} = 0 \text{ and } \det r \equiv R^{13}R^{24} - R^{14}R^{23} = 0 \tag{L.24}$$

where we have introduced the 2×2 matrix:

$$r^{AA'} = \begin{bmatrix} R^{14} & -R^{13} \\ R^{24} & -R^{23} \end{bmatrix}. \tag{L.25}$$

To deduce which points of twistor space are to be deleted, it is helpful to write out the incidence relation matrix equation in terms of two component spinors. In this case, writing $(Z^1, Z^2, Z^3, Z^4) = (\omega^1, \omega^2, \pi_{1'}, \pi_{2'})$, we have:

$$R^{34}\omega^A = r^{AA'}\pi_{A'} \tag{L.26}$$

$$R^{12}\pi_{A'} = \left(\varepsilon^{-1} \cdot r \cdot \varepsilon\right)^T_{A'A} \omega^A \tag{L.27}$$

where the matrix ε is:

$$\varepsilon = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}. \tag{L.28}$$

Let us first consider the twistor lift of i^∞. This is a single point of $\mathbb{RM}^\#$, and so lifts to a $\mathbb{CP}^1 \subset \mathbb{PT}$. The point of tangency is defined by the conditions $R^{34} = 0$ and $r = 0$. In other words, equation (L.26) is trivially satisfied and equation (L.27) reduces to:

$$R^{12}\pi_{A'} = 0. \tag{L.29}$$

Since all of the other $R^{\alpha\beta}$ vanish at this point and we are in projective space, $R^{12} \neq 0$ and we must require $\pi_{1'} = 0$ and $\pi_{2'} = 0$. We denote this distinguished $\mathbb{CP}^1$ as $\mathbb{CP}^1(\infty)$:

$$\mathbb{CP}^1(\infty) = \{\pi_{1'} = \pi_{2'} = 0\} \tag{L.30}$$

Next consider the twistor lift of the remaining points at infinity, namely $\mathscr{I}^+$ and $\mathscr{I}^-$. These are defined as all points on the quadric such that $R^{34} = 0$, but $r \neq 0$. Returning to the quadric equation, this enforces the condition that r is a non-zero matrix but with $\det r = 0$.

Such 2×2 matrices can be written as an outer product:

$$r^{AA'} = \zeta^A \rho^{A'} \tag{L.31}$$

for some ζ^A and $\rho^{A'}$. The incidence relations now reduce to:[118]

$$[\rho, \pi] = 0 \tag{L.32}$$
$$R^{12}\pi_{A'} = \rho_{A'} \langle \zeta, \omega \rangle \, . \tag{L.33}$$

So, the two component vector $\pi_{A'}$ is parallel to $\rho_{A'}$. Hence, each point of $\mathscr{I}^+$ and $\mathscr{I}^-$ lifts to a $\mathbb{CP}^1$ which intersects $\mathbb{CP}^1(\infty)$.

Let us now turn to the characterization of the points which are not at infinity in Minkowski space so that $R^{34} \neq 0$. In this chart, it is helpful to introduce a 2×2 position matrix $x^{AA'}$ with entries:

$$x^{AA'} = \frac{1}{iR^{34}} r^{AA'} = \frac{1}{i} \begin{bmatrix} R^{14}/R^{34} & -R^{13}/R^{34} \\ R^{24}/R^{34} & -R^{23}/R^{34} \end{bmatrix} . \tag{L.34}$$

In this patch, Minkowski space is represented as the paraboloid:

$$\frac{R^{12}}{R^{34}} + \det x = 0 \tag{L.35}$$

as follows from substitution into the quadric equation. In these variables, the incidence relation is:

$$\omega^A = i x^{AA'} \pi_{A'} . \tag{L.36}$$

The limit $R^{34} \to 0$ corresponds to $x \to \infty$, which lifts to the "line at infinity".

Along the real slice of the complexified spacetime, the matrix x satisfies:

$$x^\dagger = -\frac{1}{i}\frac{1}{\overline{R^{34}}} \begin{bmatrix} \overline{R^{14}} & \overline{R^{24}} \\ -\overline{R^{13}} & -\overline{R^{23}} \end{bmatrix} = \frac{1}{i}\frac{1}{R^{34}} \begin{bmatrix} R^{14} & -R^{13} \\ R^{24} & -R^{23} \end{bmatrix} = x. \tag{L.37}$$

L.2 Anti de Sitter Space

Let us now turn to a similar characterization of Anti de Sitter space. Topologically, Anti de Sitter space is given by an $S^1 \times \mathbb{R}^3$, though we shall work with the covering space, which is topologically equivalent to $\mathbb{R} \times \mathbb{R}^3$. The points at infinity in $\mathbb{RM}^\#$ are those along the real quadric with $W = 0$. The Penrose diagram is quite different from Minkowski space. In this case, it is given by an infinitely long strip (it is also helpful to view it as an infinitely long cylinder). The line on the left is conventionally defined to denote the center of AdS, while the line at the right defines the "boundary" of AdS space. Spacelike infinity i^0 and

[118] It is customary in the scattering amplitudes literature to denote symplectic pairings (via the appropriate ε tensor) of left- and right-handed spinors as $\langle \bullet, \bullet \rangle$ and $[\bullet, \bullet]$.

null infinity $\mathscr{I}$ are given by the same timelike three-surface, which we denote by $\mathscr{I}$. The subspace $\mathscr{I}$ has topology $S^2 \times \mathbb{R}$. Timelike future infinity i^+ and past infinity i^- correspond to two points disjoint from $\mathscr{I}$ and sit midway between the center of AdS and $\mathscr{I}$, infinitely far in the past and future of the infinite strip.

In this case we observe that along the hyperplane $W = 0$, there is no distinguished point which shares the same tangent space vectors as the quadric. This already means that there is no "distinguished" spacelike point at infinity, as in the case of Minkowski space. The locus of points to be deleted from $\mathbb{CM}^{\#}$ satisfy:

$$\text{Delete: } I^{(AdS)}_{\alpha\beta} R^{\alpha\beta} = R^{12} - \frac{1}{2} R^{34} = 0. \tag{L.38}$$

The quadric equation now reduces to:

$$R^{12} - \frac{1}{2} R^{34} = 0 \text{ and } \frac{1}{2}\left(R^{34}\right)^2 + \det r = 0 \tag{L.39}$$

where the 2×2 matrix r is the same as in the case of Minkowski space:

$$r = \left[\begin{array}{cc} R^{14} & -R^{13} \\ R^{24} & -R^{23} \end{array}\right]. \tag{L.40}$$

The zero set of equation (L.39) consists of the branches $R^{34} = R^{12} = 0$, while the other has $R^{34}, R^{12} \neq 0$.

Along the branch $R^{12} = R^{34} = 0$, we have $\det r = 0$, with $r \neq 0$. Hence, the matrix r can be written as the outer product:

$$r^{AA'} = \zeta^A \rho^{A'}. \tag{L.41}$$

The incidence relation then reduces to the two spinor equations:

$$[\rho, \pi] = \langle \zeta, \omega \rangle = 0. \tag{L.42}$$

Next consider the locus of points with $R^{34} \neq 0$. By the projective scaling symmetry, it is enough to set $R^{12} = 1$, $R^{34} = 2$. The matrix r then satisfies $\det r = 2$, and so in particular is invertible. Since only two of the incidence relations are independent, it is enough to consider the incidence relation:

$$\left[\begin{array}{c} \omega^1 \\ \omega^2 \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc} R^{14} & -R^{13} \\ R^{24} & -R^{23} \end{array}\right] \cdot \left[\begin{array}{c} \pi_{1'} \\ \pi_{2'} \end{array}\right] \tag{L.43}$$

where $R^{24}R^{13} - R^{13}R^{24} = 2$.

Let us next turn to the incidence relation for points not at conformal infinity. The admissable points on the quadric satisfy:

$$R^{12} - \frac{1}{2}R^{34} = l. \tag{L.44}$$

In this case, the quadric equation becomes:

$$\frac{1}{2}\left(R^{34}\right)^2 + l \cdot R^{34} + \det r = 0. \tag{L.45}$$

Solving for R^{34} and R^{12} then yields:

$$R^{34} = -l \pm \sqrt{l^2 - 2\det r} \tag{L.46}$$

$$R^{12} = \frac{1}{2}\left(l \pm \sqrt{l^2 - 2\det r}\right) \tag{L.47}$$

where both signs are admissable. When $l^2 > 2\det r$, the incidence relations can then be written as:

$$\omega^A = \frac{-r^{AA'}\pi_{A'}}{l \mp \sqrt{l^2 - 2\det r}} \tag{L.48}$$

$$\pi_{A'} = \frac{2\left(\varepsilon^{-1} \cdot r \cdot \varepsilon\right)^T_{A'A}\omega^A}{l \pm \sqrt{l^2 - 2\det r}} \tag{L.49}$$

for $l^2 - 2\det r > 0$. Note that when one denominator vanishes, the other remains finite.

L.3 De Sitter Space

Finally, we turn to de Sitter space. De Sitter space is specified by the slice through $\mathbb{RM}^\#$ which satisfies $T = l$. In other words, we are instructed to delete the subspace $T = 0$. The Penrose diagram of de Sitter space is a square, but with sides parallel to the page. This spacetime has topology $\mathbb{R} \times S^3$. The left edge of the Penrose diagram defines the north pole of the S^3, and the right edge defines the south pole. The bottom edge corresponds to past null infinity $\mathscr{I}^-$, and the upper edge corresponds to future null infinity $\mathscr{I}^+$. See figure 15 for a depiction of the Penrose diagram.

The locus of points to be deleted from $\mathbb{CM}^\#$ satisfy:

$$\text{Delete: } I^{(dS)}_{\alpha\beta}R^{\alpha\beta} = \frac{i}{\sqrt{2}}\left(R^{23} - R^{14}\right) = 0. \tag{L.50}$$

Along this locus, the quadric reduces to:

$$\left(R^{14}\right)^2 + \det r = 0 \tag{L.51}$$

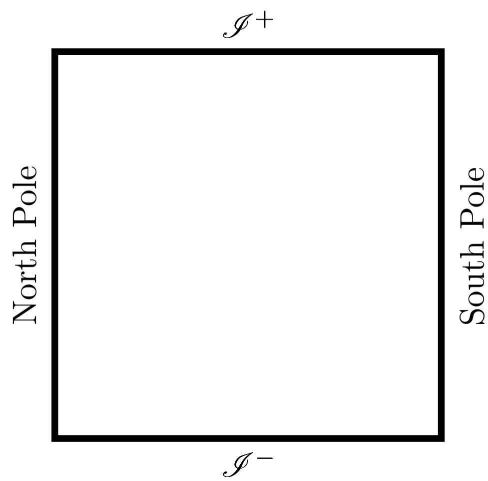

Figure 15: Penrose diagram for 4D de Sitter space. Each interior point denotes an S^2, and each horizontal line denotes an S^3. The north pole of the S^3 is specified on the left and the south pole of the S^3 is specified on the right. Future null infinity is at $\mathscr{I}^+$, and past null infinity is at $\mathscr{I}^-$. A signal sent from the south pole to the north pole takes an infinity amount of time (bottom right to upper left vertices of the square).

where as opposed to the case of Minkowski space and AdS, here the matrix r is given by:

$$r = \begin{bmatrix} R^{12} & R^{24} \\ R^{13} & R^{34} \end{bmatrix}. \tag{L.52}$$

The incidence relations can be expressed as:

$$R^{14}\mu^a = r^{aa'}\lambda_{a'} \tag{L.53}$$

$$R^{23}\lambda_{a'} = \left(\varepsilon^{-1}\cdot r\cdot\varepsilon\right)^T_{a'a}\mu^a \tag{L.54}$$

where we have introduced a different basis of spinors $\lambda_{a'}$ and μ^a:

$$\begin{bmatrix} \lambda_{1'} \\ \lambda_{2'} \end{bmatrix} = \begin{bmatrix} Z^4 \\ Z^1 \end{bmatrix}, \quad \begin{bmatrix} \mu^1 \\ \mu^2 \end{bmatrix} = \begin{bmatrix} Z^2 \\ Z^3 \end{bmatrix}. \tag{L.55}$$

Let us now study the locus of deleted points. Consider first the branch $R^{14} = R^{23} = 0$ and $\det r = 0$. In terms of the original real variables, this condition is:

$$2\det r = V^2 - W^2 - X^2 - Y^2 = 0. \tag{L.56}$$

Due to the projective condition, r is non-zero, and so can be represented as the outer product:

$$r^{aa'} = \phi^a \chi^{a'}. \tag{L.57}$$

The corresponding twistor line is then:

$$\phi^a \mu_a = \chi^{a'} \lambda_{a'} = 0. \tag{L.58}$$

Next consider the locus $R^{14} = R^{23} \neq 0$. In this case, we have $\det r \neq 0$, and the incidence relation becomes:

$$\mu^a = \frac{r^{aa'} \lambda_{a'}}{R^{14}}. \tag{L.59}$$

We now turn to the incidence relation for the points which are a part of dS. For these points, we have $R^{23} = R^{14} - i\sqrt{2}l$. The quadric reduces to:

$$\left(R^{14}\right)^2 - i\sqrt{2}l \cdot R^{14} + \det r = 0 \tag{L.60}$$

so that R^{14} and R^{23} satisfy:

$$R^{14} = i\frac{l \pm \sqrt{l^2 + 2\det r}}{\sqrt{2}} \tag{L.61}$$

$$R^{23} = -i\frac{l \mp \sqrt{l^2 + 2\det r}}{\sqrt{2}}. \tag{L.62}$$

The incidence relations can now be written as:

$$\mu^a = \frac{-i\sqrt{2} r^{aa'} \lambda_{a'}}{l \pm \sqrt{l^2 + 2\det r}} \tag{L.63}$$

$$\lambda_{a'} = \frac{i\sqrt{2}(\varepsilon^{-1} \cdot r \cdot \varepsilon)^T_{a'a} \mu^a}{l \mp \sqrt{l^2 + 2\det r}}. \tag{L.64}$$

L.3.1 Observers on the S^3

To conclude our discussion, we consider the twistor associated with an observer sitting at the north and south poles of the S^3. With respect to the coordinate system satisfying the quadric equation:

$$-V^2 + W^2 + X^2 + Y^2 + Z^2 = l^2, \tag{L.65}$$

the north and south poles are taken to be situated at $X = Y = Z = 0$. By convention, we take:

$$W_{north} = +\sqrt{l^2 + V^2} \tag{L.66}$$

$$W_{south} = -\sqrt{l^2 + V^2}. \tag{L.67}$$

At finite times in global coordinates.

Let us now consider in more detail the twistor line for the north and south poles at infinity. This is given by $X = Y = Z = 0$ and $V = \pm W$. In terms of the R coordinates, this is the point:

$$N_\infty\text{: } R^{12} \neq 0 \text{ all other } R^{\alpha\beta} = 0. \tag{L.68}$$

$$S_\infty\text{: } R^{34} \neq 0 \text{ all other } R^{\alpha\beta} = 0. \tag{L.69}$$

Plugging these values into the incidence relations, the north pole and south pole of the S^3 are respectively given by:

$$\mathbb{CP}^1_N(\infty) : \left\{Z^3 = Z^4 = 0\right\} \tag{L.70}$$

$$\mathbb{CP}^1_S(\infty) : \left\{Z^1 = Z^2 = 0\right\}. \tag{L.71}$$

A very interesting feature of these two $\mathbb{CP}^1$'s is that they do not intersect. This is simply the twistorial lift of the statement that a signal sent from the south pole takes infinite time to reach the north pole.

Next consider the twistor lift of the south pole at finite global times. Such points satisfy $X = Y = Z = 0$ and $T = l$. Hence, $W_{north} = +\sqrt{l^2 + V^2}$ and $W_{south} = -\sqrt{l^2 + V^2}$. In terms of the R coordinates we have:

$$R^{12} = \frac{1}{2}\left(V \pm \sqrt{l^2 + V^2}\right),\ R^{13} = 0,\ R^{14} = \frac{il}{\sqrt{2}} \tag{L.72}$$

$$R^{23} = -\frac{il}{\sqrt{2}},\ R^{24} = 0,\ R^{34} = V \mp \sqrt{l^2 + V^2}. \tag{L.73}$$

Solving the incidence relations, the corresponding twistor lines are, as a function of V, given by:

$$\mathbb{CP}^1_N(V) : \left\{\left(V + \sqrt{l^2 + V^2}\right) Z^2 + i\sqrt{2}lZ^4 = \left(V + \sqrt{l^2 + V^2}\right) Z^1 + i\sqrt{2}lZ^3 = 0\right\} \tag{L.74}$$

$$\mathbb{CP}^1_S(V) : \left\{\left(V - \sqrt{l^2 + V^2}\right) Z^2 - i\sqrt{2}lZ^4 = \left(V - \sqrt{l^2 + V^2}\right) Z^1 - i\sqrt{2}lZ^3 = 0\right\}. \tag{L.75}$$

Let us note that sending $V \to -V$ interchanges the two twistor lines.

In the limit of infinite V, we obtain:

$$\mathbb{CP}^1_N(V \to \infty) : \left\{Z^1 = Z^2 = 0\right\} = \mathbb{CP}^1_N(\infty) \tag{L.76}$$

$$\mathbb{CP}^1_S(V \to \infty) : \left\{Z^3 = Z^4 = 0\right\} = \mathbb{CP}^1_S(\infty). \tag{L.77}$$

M Alternative Supersymmetric Action

In the main body of this note we discussed a physically motivated choice for Frobenius conjugation on fermionic fields. In this Appendix we briefly discuss the structure of the "other choice" where we instead enforce the condition:

$$F(\chi\psi) = F(\chi)F(\psi) = \chi\psi, \tag{M.1}$$

that is, we *do not* reverse the order of multiplication for products of fermion after Frobenius conjugation. This leads to some algebraic simplifications in the construction of various actions. The price we pay, however, is that there are now some new minus sign factors which must be taken into account.

We now construct an example of a 1D supersymmetric action in characteristic p, but in which our fermion products are invariant under the Frobenius automorphism. With this in mind, we now consider a single $\mathbb{F}_p$ valued bosonic field $\phi(t)$ and a pair of $\mathbb{F}_p$ valued Grassmann variables $\chi(t)$ and $\psi(t)$. We also introduce an $\mathbb{F}_p$ valued auxiliary field $f(t)$ and a superpotential $W(\phi)$ which will be a polynomial in the ϕ variable with coefficients in $\mathbb{F}_p$. We denote the derivatives of W with respect to ϕ as W' and W''. Our Lagrangian is:

$$L = \frac{1}{2}(\partial_t\phi)^2 + \chi\partial_t\psi - \frac{1}{2}f^2 + W'f + W''\chi\psi. \tag{M.2}$$

Observe that there are no factors of "i" and the products of fermions are invariant under Frobenius conjugation. An important comment is that the sign of the quadratic term of the auxiliary field has flipped sign compared with our treatment in the main body. This does not really mean the theory has a problematic potential since one could instead write $-1 = (p-1)$. Indeed, we have already mentioned that notions such as the signature of a "metric" lose their meaning in characteristic p anyway.

We now verify that this Lagrangian is supersymmetric. We introduce the two variations:

$$\delta_1\phi = \psi, \quad \delta_1\psi = 0, \quad \delta_1\chi = -(\partial_t\phi + f), \quad \delta_1 f = -\partial_t\psi \tag{M.3}$$

$$\delta_2\phi = \chi, \quad \delta_2\psi = -(\partial_t\phi - f), \quad \delta_2\chi = 0, \quad \delta_2 f = +\partial_t\chi. \tag{M.4}$$

Consider first varying with respect to δ_1. This yields:

$$\delta_1 L = (\partial_t\phi)(\partial_t\psi) + (-\partial_t\phi - f)\partial_t\psi - (-\partial_t\psi)f \tag{M.5}$$

$$+ W''(\psi)f + W'(-\partial_t\psi) \tag{M.6}$$

$$+ W''(-\partial_t\phi - f)\psi \tag{M.7}$$

$$= \partial_t(-W'\psi). \tag{M.8}$$

Observe that we have a "total derivative," which as we already mentioned, will be dropped (since it specifies an exact differential form).

Next, consider varying with respect to δ_2. This yields:

$$\begin{aligned}\delta_2 L = & (\partial_t \phi)\,(\partial_t \chi) - \partial_t(-\partial_t \phi + f)\chi - (+\partial_t \chi) f && \text{(M.9)}\\ & + W''(\chi) f + W'(+\partial_t \chi) && \text{(M.10)}\\ & - W''(-\partial_t \phi + f)\chi && \text{(M.11)}\\ & = \partial_t((\partial_t \phi)\,\chi - f\chi + W'\chi), && \text{(M.12)}\end{aligned}$$

which is again a "total derivative." Integrating out the auxiliary field f, we arrive at a potential for the field ϕ given by:

$$V(\phi) = -\frac{1}{2} W' W', \tag{M.13}$$

which has a sign flip relative to the characteristic zero case. This is in some sense immaterial because "positive and negative" have little meaning in the characteristic p setting. For example, we could view the "negative number" $-1 = p - 1$ as actually a "positive number."

N Evidence for Quantized FI Parameters

In this Appendix we present some evidence that quantization of FI parameters is compatible with string theory considerations. This issue has been studied using the formalism of 4D $\mathcal{N} = 1$ supergravity in references [15–17]. These considerations do not constitute a full construction, and amount to consistency conditions which would be needed in order to make sense of any putative effective field theory. Indeed, because the size of these quantized parameters is near the Planck scale, effective field theory arguments are not fully justified. Our analysis will be similarly limited since we will be making use of notions from effective field theory, but applying them in a regime where mass scales are extremely large. Nevertheless, we find it encouraging that this analysis is compatible with such considerations.

The main idea will be to consider the $U(1)$ gauge theory associated with the worldvolume of a probe D3-brane in type IIB string theory on a spacetime of the form $M_4 \times M_6$ so that M_4 refers to the macroscopic spacetime and M_6 to the small internal directions. As the subscripts suggest, M_4 is taken to be a four-manifold and M_6 is taken to be a six-manifold. The configuration of branes we consider consists of a D9- / anti-D9-brane pair, and a D7-brane filling M_4 and wrapping an internal four-cycle S_{GUT}. We assume that S_{GUT} is contractible and is threaded by non-zero NS two-form flux and that the volume of M_6 is suitably quantized in flux units as set by the D9- / anti-D9-brane pair.

In the decoupling limit of the D7-brane gauge theory, this leads to a non-commutative gauge theory on the internal directions of S_{GUT}, as in references [335, 283]. On the Higgs branch of the D3-brane probe theory, the D3-brane dissolves as an instanton, which corresponds to an anti-self-dual field strength in the internal directions of M_4. We assume that the B-flux has been chosen so that it is self-dual. In other words as explained in [336], the D3-brane sees a background of anti-D3-branes. In this case, the small instanton limit is absent, and there is instead an FI parameter ξ in the probe D3-brane, which is set by the value of the B-field.

Our first task is to estimate the value of ξ. At the origin of moduli space, the energy density for this system is:

$$E = \frac{g_{\mathrm{YM}}^2}{2}\xi^2. \tag{N.1}$$

We can also attempt to evaluate this directly in the brane configuration: The energy density is that of a D3- / anti-D3-brane annihilation. Our admittedly crude estimate of this will be to simply sum the tensions for a D3-brane and anti-D3-brane, which should really be viewed as a lower bound:[119]

$$E = T_{\mathrm{D3}} + T_{\overline{\mathrm{D3}}} = 2 \times \frac{1}{g_s}\frac{1}{(2\pi)^3}\frac{1}{(\alpha')^2}. \tag{N.2}$$

The gauge coupling for a single D3-brane is also fixed by the DBI action to be $g_{\mathrm{YM}}^2 = 2\pi g_s$.

[119] We provide a mild "improvement" on this estimate in section N.1.

We therefore obtain our value for ξ:

$$\xi = 2 \times \frac{1}{g_s} \frac{1}{(2\pi)^2} \frac{1}{\alpha'}. \tag{N.3}$$

We now relate this value to the 4D reduced Planck mass. Recall that the 10D Newton's constant is given by:

$$16\pi G_N^{(10D)} = (2\pi)^7 g_s^2 (\alpha')^4. \tag{N.4}$$

Compactifying on a six-manifold, we obtain the 4D Newton's constant:

$$16\pi G_N^{(4D)} = \frac{(2\pi)^7 g_s^2 (\alpha')^4}{\mathrm{Vol}(M_6)}. \tag{N.5}$$

Now, in the present setup with a D9- / anti-D9-brane pair, we can compute the volume $\mathrm{Vol}(M_6)$ in units associated with switching on a non-trivial flux in the M_4 directions which induces a Euclidean D5- / anti-D5-brane pair wrapped over M_6. Hence, the natural scaling of $\mathrm{Vol}(M_6)$ is set in units of D5-brane tension:

$$\frac{1}{\mathrm{Vol}(M_6)} = N_{\mathrm{D5}} \left(T_{\mathrm{D5}} + T_{\overline{\mathrm{D5}}}\right) = 2N_{D5} \times \frac{1}{g_s} \frac{1}{(2\pi)^5} \frac{1}{(\alpha')^3}, \tag{N.6}$$

where N_{D5} is a positive integer. Plugging in, we obtain the value of the 4D Newton's constant:

$$16\pi G_N^{(4D)} = 2N_{\mathrm{D5}} \times (2\pi)^2 g_s(\alpha') = 2N_{\mathrm{D5}} \times \frac{2}{\xi} \tag{N.7}$$

or:

$$\xi = N_{\mathrm{D5}} \times \frac{4}{16\pi G_N^{(4D)}} = 2N_{\mathrm{D5}} \times M_{\mathrm{pl}}^2 \tag{N.8}$$

where $M_{\mathrm{pl}}^2 = 1/8\pi G_N^{(4D)}$. We note that it is appropriate to consider a D5-brane background rather than some other (p,q) five-brane because in the duality frame being considered, D5-branes are the lowest tension five-branes available.

Let us now generalize this to FI parameters of a d-dimensional gauge theory, for $d = 2k$. In this case, a similar argument yields the relations:

$$16\pi G_N^{(d)} = g_s \times 2N_{\mathrm{D(9-d)}} \times \frac{1}{(2\pi)^{2-d}} \frac{1}{(\alpha')^{(2-d)/2}} \tag{N.9}$$

$$\xi^2 = 4 \times T_{(d-1)}^2 \times (2\pi\alpha')^2 = 4 \times \frac{1}{g_s^2} \times \frac{1}{(2\pi)^{2d-4}} \times \frac{1}{(\alpha')^{(d-2)}} \tag{N.10}$$

which in turn implies:

$$\xi = 2 \times \frac{1}{g_s} \times \frac{1}{(2\pi)^{d-2}} \times \frac{1}{(\alpha')^{(d-2)/2}} \tag{N.11}$$

or:

$$\xi = 2N_{D(9-d)} \times \frac{1}{8\pi G_N^{(d)}}. \tag{N.12}$$

So, we see that the quantization of ξ is again in even steps of $1/8\pi G_N^{(d)}$.

N.1 Issues and Workarounds

The main weakness in this line of argument is that we are working in the limit of large field ranges, and so cannot really rely on effective field theory reasoning, as we have implicitly done in matching parameters of the supersymmetric effective field theory of a D3-brane. An additional concern is that to estimate various energy densities, we made the crude approximation given by summing the individual tensions of the branes, with little regard for the formation of non-perturbative bound states. That being said, the estimate is "not as bad" as one might initially think. To see why, suppose that we return to line (N.2) and allow for an overall constant which parameterizes our ignorance of brane annihilation:

$$E = \varepsilon_{\mathrm{D3}} \times (T_{\mathrm{D3}} + T_{\overline{\mathrm{D3}}}) = \varepsilon_{\mathrm{D3}} \times 2 \times \frac{1}{g_s} \frac{1}{(2\pi)^3} \frac{1}{(\alpha')^2}. \tag{N.13}$$

We would then get a different value for the FI parameter:

$$\xi = \sqrt{\varepsilon_{\mathrm{D3}}} \times 2 \times \frac{1}{g_s} \frac{1}{(2\pi)^2} \frac{1}{\alpha'}. \tag{N.14}$$

Let us further assume that in our volume estimate based on D5- / anti-D5 annihilation, a similar constant appears. Then, returning to line (N.6) would yield:

$$\frac{1}{\mathrm{Vol}(M_6)} = \varepsilon_{\mathrm{D5}} \times N_{\mathrm{D5}} \left(T_{D5} + T_{\overline{\mathrm{D5}}}\right) = \varepsilon_{\mathrm{D5}} \times 2N_{\mathrm{D5}} \times \frac{1}{g_s} \frac{1}{(2\pi)^5} \frac{1}{(\alpha')^3}. \tag{N.15}$$

Recomputing the relation between the 4D Newton's constant and the FI Parameter now yields:

$$16\pi G_N^{(4D)} = \frac{\varepsilon_{\mathrm{D5}}}{\sqrt{\varepsilon_{\mathrm{D3}}}} \times 2N_{\mathrm{D5}} \times (2\pi)^2 g_s(\alpha') = \frac{\varepsilon_{\mathrm{D5}}}{\sqrt{\varepsilon_{\mathrm{D3}}}} \times 2N_{\mathrm{D5}} \times \frac{2}{\xi}, \tag{N.16}$$

i.e., to have quantization of the FI parameter, we would need the following relation to hold:

$$\frac{\varepsilon_{\mathrm{D5}}}{\sqrt{\varepsilon_{\mathrm{D3}}}} \times N_{\mathrm{D5}} \in \mathbb{Z}. \tag{N.17}$$

It is unclear to us whether this is satisfied, but if it holds, then one can proceed as before (where we set the ε's to one). While this is still quite speculative, we find it encouraging. We defer these issues to future investigations.

O Brief Review of $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$

In this Appendix we present a brief review of the p-adic numbers, $\mathbb{Q}_p$.[120] We also discuss the related fields $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$. This material is entirely standard, and can be found in many places. For an exposition which is "physicist friendly", see for example [12] and references therein.

Now, we have already given one definition of this number system by first constructing the ring of integers $\mathbb{Z}_p$ as the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$, namely:

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}. \tag{O.1}$$

One can then define $\mathbb{Q}_p$ as the field of fractions for $\mathbb{Z}_p$, with induced topology set by the inverse limit system.

A perhaps more direct way to define the p-adic numbers is to begin with the rational numbers $\mathbb{Q}$ and introduce a corresponding p-adic norm $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ by the rule that for $x = a/b$ a rational number with a and b integers such that $a = a_0 p^m$ and $b = b_0 p^n$ with a_0 and b_0 both relatively prime to p, the p-adic norm of x is given by:

$$|x|_p = p^{n-m}. \tag{O.2}$$

The p-adic norm satisfies the following properties:

$$|x|_p \geq 0 \quad \text{for all } x \tag{O.3}$$

$$|x|_p = 0 \quad \text{if and only if } x = 0 \tag{O.4}$$

$$|x|_p \, |y|_p = |xy|_p \quad \text{for all } x \text{ and } y \tag{O.5}$$

$$|x+y|_p \leq \max\left(|x|_p \, , |y|_p\right) \quad \text{for all } x \text{ and } y. \tag{O.6}$$

The last condition is known as the "strong triangle equality", and from it, one can derive the standard triangle inequality $|x+y|_p \leq |x|_p + |y|_p$. The strong triangle inequality defines what is referred to as a non-Archimedean norm. It is also customary to interpret the standard Archimedean norm as being obtained from "the prime at $p = \infty$".

The topology defined by the p-adic norm is rather different from the Archimedean case. For example, consider any series of the form:

$$a = p^m \sum_{j \geq 0} a_j p^j, \tag{O.7}$$

with $|a_j|_p = 1$ for all j. This converges, and has p-adic norm p^{-m}. An especially important

[120] To our knowledge, the p-adic numbers were first introduced in reference [337].

subset is the ring of integers $\mathbb{Z}_p$ as defined by the ball of unit radius:

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p \quad \text{such that} \quad |x|_p \leq 1 \right\}. \tag{O.8}$$

Starting from $\mathbb{Z}_p$, observe that there is a maximal prime ideal generate by p itself. Taking the quotient $\mathbb{Z}_p/p\mathbb{Z}_p$ results in the residue field $\mathbb{F}_p$.

In the case of the real numbers $\mathbb{R}$, the algebraic closure is given by the complex numbers $\mathbb{C}$, which is also metrically complete. For the p-adics, the separable algebraic closure $\overline{\mathbb{Q}}_p$ also has a p-adic norm, which can be defined by extension of the one defined over $\mathbb{Q}_p$. For a Galois extension K of $\mathbb{Q}_p$, we can treat K as a vector space over $\mathbb{Q}_p$. In particular, for any $\alpha \in K$, we can consider the product over all the Galois conjugates. Call this $\mathrm{Norm}_{K/\mathbb{Q}_p}(\alpha)$. By construction, $\mathrm{Norm}_{K/\mathbb{Q}_p}(\alpha)$ is an element of $\mathbb{Q}_p$, and so we can also take its p-adic norm. The extension of the p-adic norm to K is then given by taking:

$$|\alpha|_p \equiv \left|\mathrm{Norm}_{K/\mathbb{Q}_p}(\alpha)\right|_p^{1/n}, \tag{O.9}$$

where $n = [K : \mathbb{Q}_p]$ is the degree of the field extension. As a point of notation, we shall often denote by $\mathbb{Q}_q$ the unramified field extension of degree n, with $q = p^n$.[121]

Proceeding to $\overline{\mathbb{Q}}_p$, the algebraic completion of $\mathbb{Q}_p$, one can show that $\overline{\mathbb{Q}}_p$ is not metrically complete. Indeed, one can construct Cauchy sequences which do not converge in $\overline{\mathbb{Q}}_p$. Following [338], one way to establish this is to apply the Baire category theorem [339], which tells us that every metrically complete space is a Baire space.[122] To establish the claim, it is enough to show that $\overline{\mathbb{Q}}_p$ is not a Baire space. Here, we have the fact that the algebraic closure has countably infinite dimension. Moreover, for every fixed $d \in \mathbb{N}$, there are a finite number of degree d field extensions of $\mathbb{Q}_p$. As a consequence, $\overline{\mathbb{Q}}_p$ cannot be metrically complete. We note that the argument fails (as it must) in the case of $\mathbb{Q}_\infty = \mathbb{R}$ since the algebraic closure is a degree two extension of $\mathbb{R}$, namely $\mathbb{C} = \mathbb{R}(\sqrt{-1})$. A perhaps more direct way to establish the same claim is to consider the collection of partial sums:

$$a_K = \sum_{n=1}^{K} p^{\left(n+\frac{1}{n}\right)}. \tag{O.10}$$

We observe that for a fixed K, the corresponding a_K resides in a finite field extension of $\mathbb{Q}_p$, namely we adjoin numbers such as $p^{1/n}$ for $n = 1, ..., K$. On the other hand, the limit $K \to \infty$ does not converge in $\overline{\mathbb{Q}}_p$.

The metric completion of $\overline{\mathbb{Q}}_p$ is known as $\mathbb{C}_p$, and this space is both algebraically closed and metrically complete. A helpful comment here (especially when we turn to the p-adic log-

[121] For a brief account of ramification, see Appendix S. The unramified extension has the pleasant feature that the residue field for $\mathbb{Q}_q$ is just $\mathbb{F}_q$.

[122] Recall that a Baire space is one for which for every countable set of dense open sets U_n for $n \in \mathbb{N}$, the intersection over all the U_n is still dense.

arithm) is that $\mathbb{C}_p$ admits a decomposition as follows (we follow the discussion and notation in [340]):

$$\mathbb{C}_p = \{p^r wu \text{ with } r \in \mathbb{Q}, \ w \in W \ \ u \in U\} = p^{\mathbb{Q}} \times W \times U, \tag{O.11}$$

where $p^{\mathbb{Q}}$ denotes all rational powers of p, W is the group of all roots of unity in $\mathbb{C}_p$, and U is the disk centered at 1 of unit radius:

$$U = \{|u-1|_p < 1 \text{ with } u \in \mathbb{C}_p\}. \tag{O.12}$$

One can also construct non-canonical isomorphisms between $\mathbb{C}_p$, $\mathbb{C}$, and thus also between $\mathbb{C}_p$ and $\mathbb{C}_{p'}$ for p and p' distinct primes. As explained in footnote 74, this follows from the fact that $\mathbb{C}$ and $\mathbb{C}_p$ have the same degree of transcendence over $\mathbb{Q}$. A non-trivial consequence of this fact is that we can also extend the p-adic norm to the complex numbers (see e.g., [341, 257]). Indeed, given a field isomorphism:

$$\phi : \mathbb{C} \to \mathbb{C}_p \tag{O.13}$$

we can assign a p-adic norm to elements $z \in \mathbb{C}$ as follows:

$$|z|_p \equiv |\phi(z)|_p. \tag{O.14}$$

By the same reasoning, the converse also holds; we can assign Archimedean norms to elements of $\mathbb{C}_p$. The use of this in the physical setting is more elusive; one is often considered with various analytic properties, and the map ϕ is not even continuous. That being said, if one continues to work at the level of algebraic structures and morphisms (as we have been advocating throughout this note) then many structures carry through.

P Witt Vectors

In this Appendix we briefly discuss some aspects of Witt vectors [342]. We saw the appearance of these in our brief discussion of crystalline cohomology in subsection 13.2, and again in section 16 when we discussed reduction of an integer valued action mod N. Again, as we are mathematical dilettantes, we will content ourselves to closely follow reference [343]. For additional discussion of Witt vectors and their applications, see reference [344].

Given a prime number p and a commutative ring R, we denote a Witt vector as $(X_0, ..., X_m, ...)$ with $X_i \in R$. Next, introduce the Witt polynomials:

$$V^{(n)} = \sum_{i=0}^{n} p^i F^i(X_i), \tag{P.1}$$

where $F(X) = X^p$ is the Frobenius morphism. We define a ring of Witt vectors on the $V^{(n)}$'s. These are also known as the "ghost components"There is an essentially unique way to make the space of Witt vectors into a commutative ring such that addition and multiplication occur componentwise. In terms of two Witt vectors U and V, we have:

$$(U + V)^{(i)} = U^{(i)} + V^{(i)} \tag{P.2}$$

$$(UV)^{(i)} = U^{(i)} V^{(i)}. \tag{P.3}$$

Given two Witt vectors $(X_0, X_1, ...)$ and $(Y_0, Y_1, ...)$ the explicit formulas for the first few entries of addition and multiplication are:

$$(X_0, X_1, ...) + (Y_0, Y_1, ...) = (X_0 + Y_0, X_1 + Y_1 + (X_0^p + Y_0^p - (X_0 + Y_0)^p/p), ...) \tag{P.4}$$

$$(X_0, X_1, ...) \times (Y_0, Y_1, ...) = (X_0 Y_0, X_0 Y_1^p + X_1^p Y_1 + p X_1 Y_1, ...) \tag{P.5}$$

where the appearance of "division by p" in the addition rule is just a formal way of condensing the notation for expanding out the binomial sum (no inverse powers of p appear in the final expressions).

For the purposes of this note, the main case of interest is the special case where R actually refers to a finite field such as $\mathbb{F}_p$ or $\mathbb{F}_q$. In the case of $\mathbb{F}_p$, the ring of Witt vectors is just the p-adic integers $\mathbb{Z}_p$ written in terms of Teichmüller representatives, and in the case of $\mathbb{F}_q$ it is the unramified extension of degree n of $\mathbb{Z}_p$.[123]

Let us explain how this works in more detail for the special case of $\mathbb{Z}_p$, the p-adic ring of integers. Recall that this space is just the elements of the p-adic numbers with p-adic norm less than or equal to one. Each such element $\phi \in \mathbb{Z}_p$ can be written as a power series:

$$x = \sum_i x_i p^i, \tag{P.6}$$

[123]For a brief discussion of ramification (or lack thereof) for local fields, see Appendix S.

with $a_i \in \{0, ..., p-1\}$. Now, an undesirable feature of this expansion is that the coefficients a_i do not respect the addition and multiplication rules of the Witt vectors. To get a suitable presentation, we instead use Teichmüller representatives. These are given by 0 as well as the $p-1$ roots of unity in $\mathbb{Z}_p$. Algorithmically, we begin with a p-adic integer x as in equation (P.6) and build a new representative:

$$x = \sum_i \omega(\overline{x}_i) p^i, \tag{P.7}$$

which converges in the p-adic metric to the original sum. The algorithm for building these representatives is also straightforward, and follows from Hensel lifting / Newton's algorithm.[124]

Rather than present this in full detail, we just illustrate with an example of the algorithm in practice, closely following the exposition in [343]. The first term in the sequence is:

$$\omega(\overline{x}_0) = x_0. \tag{P.8}$$

After this, we construct $\omega(\overline{x}_1)$ by finding the unique solution of $x^{p-1} - 1 = 0 \bmod p^2$ such that $x = x_0 \bmod p$. Call this solution $\omega(\overline{x}_1)$. Next, we compute $x^{p-1} - 1 = 0 \bmod p^3$ such that $x = \omega(\overline{x}_1) \bmod p^2$. Observe that these representatives do not necessarily belong to the set $\{0, 1, ..., p-1\}$. They do, however, have the important property that $\omega(\overline{x}_i)^p = \omega(\overline{x}_i) \bmod p^{i+1}$, which is what makes them more suited to an analysis which respects Frobenius conjugation. As an example, the first few entries of the Witt vector for 2 with respect to the prime $p = 5$ are $(2, 7, 57, ...)$.

[124] This follows from Hensel's lemma (see [345, 346]). Following [347], a "basic version" of this lemma asserts that for $f(t) \in \mathbb{Z}_p[t]$ and $a \in \mathbb{Z}_p$ such that $f(a) = 0 \mod p$ and $f'(a) \neq 0 \mod p$, then there is a unique $\alpha \in \mathbb{Z}_p$ such that $f(\alpha) = 0$ in $\mathbb{Z}_p$ and $\alpha = 0 \mod p$. This can be extended in various ways to more abstract settings. See e.g., reference [347] as well as [348] for additional discussion.

Q Exponential Function on the p-adics

In this Appendix we briefly review the convergence of the p-adic exponential function. Recall that for a series such as:

$$\sum_{n\geq 0} c_n x^n \tag{Q.1}$$

the radius R of convergence is given by:

$$R = \frac{1}{\limsup\limits_{n\to\infty} |c_n|^{1/n}}. \tag{Q.2}$$

We are specifically interested in the case of the exponential function, as specified by the power series:

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}, \tag{Q.3}$$

interpreted as a sum in either $\mathbb{Q}_p$ or $\mathbb{C}_p$. In our case, it is enough to determine the values of t such that:

$$\lim_{n\to\infty} \left|\frac{x^n}{n!}\right|_p < 1. \tag{Q.4}$$

Our discussion closely follows the one given in reference [349] (see also [350]). We begin by estimating the p-adic norm for $n!$ in the large n limit, which we write as:

$$|n!|_p = p^{\mathrm{ord}_p(n!)}, \tag{Q.5}$$

where we have introduced the p-adic order of $n!$, as denoted by $\mathrm{ord}_p(n!)$. To begin, we count the number of powers of p which appear in $n!$. To this end, we observe that this can be written as:

$$\mathrm{ord}_p(n!) = \sum_{k\geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor, \tag{Q.6}$$

namely, we divide by powers of p and round down. This works because in $n!$, we are counting how many integers up to n are divisible by p, and then by p^2, and so on. With this step in place, we introduce the p-adic expansion of n:

$$n = a_0 + a_1 p + ... + a_r p^r. \tag{Q.7}$$

So, we see that the sum over k in equation (Q.6) actually ranges from $k = 1$ up to $k = r$. Moreover, we have, for $1 \leq k \leq r$:

$$\left\lfloor \frac{n}{p^k} \right\rfloor = a_k + ... + a_r p^{r-k}. \tag{Q.8}$$

Performing the sum over k and regrouping terms with common values of a_i, we have:

$$\sum_{k\geq 1}\left\lfloor\frac{n}{p^k}\right\rfloor = a_1 + a_2(1+p) + ... + a_r(1+p+...+p^{r-1}) \tag{Q.9}$$
$$= \frac{1}{p-1}\left(a_1(p-1) + a_2(p^2-1) + ... + a_r(p^r-1)\right), \tag{Q.10}$$

so in other words we get:

$$\mathrm{ord}_p(n!) = \frac{1}{p-1}\left(n - (a_0 + a_1 + ... + a_r)\right). \tag{Q.11}$$

From this, we can clearly bound $\mathrm{ord}_p(n!)$ as:

$$\frac{1}{p-1}\left(n - (r+1)(p-1)\right) \leq \mathrm{ord}_p(n!) < \frac{n}{p-1}. \tag{Q.12}$$

On the other hand, we also have that:

$$p^r \leq n < p^{r+1}, \tag{Q.13}$$

so we have:

$$r \leq \log_p n < r+1. \tag{Q.14}$$

Consequently, the inequalities of line (Q.12) can also be expressed as:

$$\frac{n}{p-1} - (\log_p n + 1) \leq \mathrm{ord}_p(n!) < \frac{n}{p-1}. \tag{Q.15}$$

Dividing by n and taking the limit $n\to\infty$, we establish:

$$\lim_{n\to\infty}\frac{\mathrm{ord}_p(n!)}{n} = \frac{1}{p-1}. \tag{Q.16}$$

Next, we return to equation (Q.4). The condition for the exponential series to converge is that:

$$\lim_{n\to\infty}\left|\frac{x}{(n!)^{1/n}}\right|_p < 1, \tag{Q.17}$$

or equivalently:

$$|x|_p < p^{-1/(p-1)}. \tag{Q.18}$$

As an additional comment, we observe that as p becomes large, we have:

$$\lim_{p\to\infty} p^{-1/(p-1)} = 1, \tag{Q.19}$$

so the radius of convergence is greatly reduced from the complex case.

R Logarithm Function on the p-adics

In the previous Appendix we discussed the exponential function on the p-adics. Here, we discuss a similar analysis in the case of the logarithm function. To begin, we recall that we have the standard power series expansion:

$$\log(1+x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n}. \tag{R.1}$$

This series converges on the p-adics for $|x|_p < 1$, as follows immediately from an application of the ratio test. Far more non-trivial is that much as in the complex case where we can define a logarithm function $\log : \mathbb{C}^\times \to \mathbb{C}$, there exists an extension of the power series to a function:[125]

$$\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p \tag{R.2}$$

which satisfies the conditions:

$$\log_p(ab) = \log_p(a) + \log_p(b) \quad \text{for} \quad a, b \in \mathbb{C}_p^\times. \tag{R.3}$$

Following [350, 351, 340], the main idea behind the extension to all of $\mathbb{C}_p^\times$ is to first write a general element $x \in \mathbb{C}_p^\times$ in the form (see equation (O.11)):

$$x = p^r w u, \tag{R.4}$$

where $r \in \mathbb{Q}$, $|w|_p = 1$, and $|u-1|_p < 1$. A canonical choice in extending the logarithm is then to require $\log_p(p) = 0$, and in this case, the evaluation of the logarithm is specified by:[126]

$$\log_p x = \log_p u. \tag{R.5}$$

This is sometimes referred to as the Iwasawa logarithm.[127]

As one might hope, the p-adic logarithm $\log_p$ behaves as the inverse function to the p-adic exponential. The only subtlety here is that we must ensure the appropriate domain of support for the various arguments to make sense. Assuming, for example, that $|x|_p < p^{-1/(p-1)}$, then we have:

$$\log_p(\exp_p x) = x, \tag{R.6}$$

and also:

$$\exp_p(\log_p(1+x)) = 1 + x. \tag{R.7}$$

[125] One notational comment: the subscript here clearly refers to working with the prime p, it is not (as in the setting over the real numbers) specifying the base of the logarithm. Indeed, shortly we will take $\log_p(p) = 0$, quite different from the "usual" case!

[126] Namely, we also require $\log_p w = 0$.

[127] It is perhaps not obvious, but one can make different choices for the extension of the logarithm function depending on the value of $\log_p(p)$. We have opted for the "simplest and canonical" choice.

In this way, we can also specify a general p-adic power:

$$x^z = \exp_p(z \log_p x), \tag{R.8}$$

provided the domains of support make sense.

The logarithm plays an important role in many areas of quantum field theory, especially in terms of non-analytic structure in scattering amplitudes. In that context, there are other generalizations which become prominent at higher order in a loop expansion. For example, one often encounters polylogarithms (see e.g., [352] and references therein):

$$\mathrm{Li}_k(x) = \sum_{n \geq 1} \frac{x^n}{n^k}, \tag{R.9}$$

where $\log(x) = -\mathrm{Li}_1(1 - x)$. There is a similar treatment of analytic continuation of the polylogarithms available, and this can also be given a geometric presentation for rigid analytic spaces using Coleman's theory of p-adic integration [353], to which we refer the interested reader for additional details.

S Ramification for Algebraic and Local Fields

In this Appendix we present a brief review of ramification theory. We begin with a brief review of the case of algebraic number fields, namely finite field extensions of $\mathbb{Q}$, and then turn to the case of local fields, where our main set of applications will involve finite field extensions of $\mathbb{Q}_p$. Again, we proceed mainly as a tourist and so will be content to provide a summary at the level given in reference [354] for algebraic number fields, and reference [355] as well as the notes of reference [356] for local fields. For a more comprehensive introduction to some of these notions, see e.g., reference [357].

To begin, suppose we have an algebraic number field K, i.e., a finite field extension of the rational numbers $\mathbb{Q}$. We can then consider the ring of integers[128] $\mathcal{O}_K$, as well as a prime ideal of $\mathcal{O}_K$, call it $\mathfrak{p}$. Now, suppose we have some finite field extension L/K. We can then consider the ring of integers $\mathcal{O}_L$, as well as the ideal generated by $\mathfrak{p}\mathcal{O}_L$. Rather importantly, it could be that inside $\mathcal{O}_L$, $\mathfrak{p}$ may have a different factorization. In general, for prime ideals $\mathfrak{p}_i$ of $\mathcal{O}_L$, we have a factorization:

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{p}_1^{a_1}...\mathfrak{p}_m^{a_m}\mathcal{O}_L, \tag{S.1}$$

and we say that the prime $\mathfrak{p}$ has ramification index a_i at prime $\mathfrak{p}_i$, whenever $a_i > 1$. We also say that the ramification is tame when all a_i are relatively prime, and otherwise we say the ramification is wild.

To give an example of ramification which we will make use of later, consider the cyclotomic extensions of $\mathbb{Q}$, as obtained by adjoining an Nth root of unity, namely solutions to $\xi^N = 1$. Call this field extension $L = \mathbb{Q}(\xi)$. In this case, we have $\mathrm{Gal}(L/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$. Taking $N = p$ a prime number, and $\ell \neq p$ some other prime number, we observe that there is no ramification over ℓ. There is, however, ramification over the prime p. To establish this, consider the roots of the polynomial $x^p - 1$. We have:

$$x^p - 1 = (x-1)(x-\xi)...(x-\xi^{p-1}). \tag{S.2}$$

Dividing both sides by $x - 1$, we get the p^{th} cyclotomic polynomial:

$$x^{p-1} + x^{p-2} + ... + 1 = (x-\xi)...(x-\xi^{p-1}). \tag{S.3}$$

Next, evaluate at $x = 1$. Then, we obtain the relation:

$$p = (1-\xi)...(1-\xi^{p-1}). \tag{S.4}$$

[128]Recall that for an algebraic number field K, the ring of integers consists of all elements in K which are also roots of a monic polynomial with integer coefficients, i.e., $x^n + c_{n-1}x^{n-1} + ... + c_0 = 0$, for some $c_i \in \mathbb{Z}$, and $n > 0$. For example, the integers $\mathbb{Z} = \mathcal{O}_{\mathbb{Q}}$ and the Gaussian integers $\mathbb{Z}[\sqrt{-1}] = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$.

Next, we observe that for $i > 1$, we have:

$$\frac{1-\xi^i}{1-\xi} = 1 + ... + \xi^{i-1}, \tag{S.5}$$

which in turn means that $1 - \xi^i$ is in the ideal generated by $(1 - \xi)$ in $\mathcal{O}_L$. Returning to equation (S.4), we conclude that since there are precisely $p - 1$ such factors we get:

$$p\mathcal{O}_L = (1 - \xi)^{p-1}\mathcal{O}_L, \tag{S.6}$$

so in other words p has ramification degree $p - 1$ in $\mathcal{O}_L$. Note that this also means $p = u(1 - \xi)^{p-1}$ for some a unit u of $\mathcal{O}_L$.[129]

Consider next the case of a local field, namely we now assume we are given K a field with a valuation $v_K : K \to \mathbb{R}$,[130] so that this valuation defines a locally compact topological field. In this Appendix we make the additional assumption that v_K is discretely valued, that is to say, for $K^\times$ the non-zero elements of K we can consider the multiplicative subgroup $v_K(K^\times) \subset \mathbb{R}^\times$, and we assume that this subgroup is generated by a single element $\pi \in v_K(K)$. It is common practice to assume that the valuation is normalized in the sense that $v_K(\pi) = 1$. Observe that the standard p-adic norm on $\mathbb{Q}_p$ with valuation $-\log_p |\bullet|$ is indeed normalized, and the uniformizer is just p.

Suppose next that we also have L a Galois extension of K.[131] The valuation extends to L, and we denote this as v_L. For both L and K, we can speak of the corresponding ring of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. In fact, an important fact is that for some $\alpha \in L$, we have the ring isomorphism $\mathcal{O}_L[\alpha] = \mathcal{O}_K$ (as follows from Hensel's lemma, see footnote 124). Correspondingly, we can construct the residue fields obtained by quotienting both $\mathcal{O}_L$ and $\mathcal{O}_K$ by their maximal prime ideals $\mathfrak{p}_L$ and $\mathfrak{p}_K$. This results in residue fields κ_L and κ_K, and in particular, κ_L is a Galois extension of κ_K. The subject of ramification theory for local fields involves measuring the possible "mismatch" between the Galois groups $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(\kappa_L/\kappa_K)$. As a first approach, we can introduce two numerical invariants which measure the appearance of possible branch cuts. We refer to the inertia degree $f_{L/K}$ as the degree of the Galois extension of κ_L/κ_K:[132]

$$f_{L/K} = [\kappa_L : \kappa_K]. \tag{S.7}$$

[129]Recall that u is a unit of a ring R provided there exists a multiplicative inverse v in R, i.e., we have $uv = vu = 1$.

[130]Recall that a valuation behaves very much like a logarithm, since for $a, b \in K$, we demand that $v(a) = \infty$ if and only if $a = 0$, and $v(ab) = v(a) + v(b)$. For example, for the p-adics $\mathbb{Q}_p$, we can take our valuation to just be $v(a) = -\log_p |p|$, in the obvious notation.

[131]Viewing L as a vector space over K, the condition that we have a Galois extension means that $\mathrm{Aut}(L/K)$ leaves K fixed.

[132]Recall that the degree of a Galois extension is just the order of the Galois group.

We also introduce the ramification index:

$$e_{L/K} = v_L(\pi_K), \tag{S.8}$$

which detects whether we are taking various roots of the original uniformizer on K in extending to L. An important property of these two numbers is that the degree of the Galois extension L/K is given by:

$$[L : K] = f_{L/K} e_{L/K}. \tag{S.9}$$

We say that an extension L/K is unramified when $e_{L/K} = 1$, namely $f_{L/K} = [L : K]$. We say that an extension L/K is totally ramified when $f_{L/K} = 1$. More group theoretically, we have the short exact sequence:

$$1 \to I_{L/K} \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(\kappa_L/\kappa_K) \to 1, \tag{S.10}$$

where $I_{L/K}$ is the "inertia group" for the field extension L/K. Observe that the field extension L/K is unramified when the inertia group is trivial.

The inertia group can be thought of as specifying the first stage of a sequence of filtrations of ramification groups. For $i \geq 0$, we consider the subgroup $G_i \subset \mathrm{Gal}(L/K)$ which leaves $\mathcal{O}_L/\mathfrak{p}_L^{i+1}$ invariant. The resulting nested sequence of normal subgroups is then given by:

$$G_{-1} \equiv \mathrm{Gal}(L/K) \supset G_0 \supset G_1 \supset ... \tag{S.11}$$

This sequence trivializes after a finite number of steps. We refer to G_0 as the inertia group, G_1 as the wild inertia group, and G_0/G_1 as the tame inertia group. In terms of these groups, the case of G_0 trivial means L/K is unramified, and the case of G_1 trivial is sometimes referred to as tamely ramified.

T Approximate Monodromy

In section 17 we discussed our physical expectations that there is a p-adic notion of monodromy which should act on the cohomology groups associated with BPS states in 4D $\mathcal{N}=2$ systems, as associated with a corresponding Seiberg-Witten curve. In this Appendix we spell out some additional details on the structure of monodromy.[133]

To be concrete, fix K a non-Archimedean local field (which we always take to be a field extension of $\mathbb{Q}_p$), and consider the family of elliptic curve given by the Tate curve:

$$E_K \equiv K^{\times}/q^{\mathbb{Z}}, \tag{T.1}$$

where here, $q \in K^{\times}$ such that $|q| < 1$, and $q^{\mathbb{Z}}$ is the subgroup of $K^{\times}$ generated by taking all powers of q.[134] We remark that this is a sensible way to construct a p-adic analog of an elliptic curve. To illustrate, we observe that in the complex analytic setting, we also get an elliptic curve if we consider $\mathbb{C}^{\times}/q^{\mathbb{Z}}$ with $q = \exp(2\pi i\tau)$, and the condition $|q| < 1$ simply means we are taking the analog of τ to be (in the complex analytic case) in the analog of the upper half-plane.

The Tate form has the desirable feature that we can present the elliptic curve in so-called Tate form:

$$y^2 + xy = x^3 + a_4 x + a_6, \tag{T.2}$$

where the coefficients a_4 and a_6 admit q-expansions (see e.g. [358]):

$$a_4 = -\sum_n 5n^3 \frac{q^n}{1-q^n} \quad \text{and} \quad a_6 = -\sum_n \frac{7n^5+5n^3}{12}\frac{q^n}{1-q^n}. \tag{T.3}$$

The j-invariant of the Tate curve can be read off in the standard fashion from its q-expansion:

$$j = q^{-1} + 744 + 196884q + ..., \tag{T.4}$$

where convergence of all sums is evaluated p-adically. Note, that the analog of "large complex structure" associated with a type I_n fiber corresponds to taking q to have small p-adic norm, for example $q = p^n$ for n very large.

We are interested in studying the action of the monodromy group action on the cohomology of the Tate curve. For illustrative purposes, we confine our discussion to ℓ-adic cohomology, though we expect a similar set of statements to hold in the case of rigid cohomology (also known as p-adic cohomology). With this in mind, let $K = \mathbb{Q}_p$ and denote by $\overline{K}$

[133]We thank A. Huang for several patient explanations. The example we present is closely based on these discussions.

[134]There is an unfortunate clash of notation with the convention of labelling finite field extensions of $\mathbb{F}_p$ as $\mathbb{F}_q$. Here, we defer to the standard usage in the theory of elliptic curves and modular forms, where we have a corresponding q-expansion. The context should make clear the sense in which we are using the variable q.

a separable algebraic closure of K. Then, we can consider the elliptic curve over the closed field L, and we can attempt to compute the cohomology group:

$$H^1(E_{\overline{K}}, \mathbb{Z}_\ell), \tag{T.5}$$

where ℓ is a prime distinct from p.

In the present context, "monodromy" involves determining the group action on $H^1(E_{\overline{K}}, \mathbb{Z}_\ell)$ from the inertia group $I_{\overline{K}/K}$ defined implicitly through the short exact sequence:

$$1 \to I_{\overline{K}/K} \to \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\kappa_{\overline{K}}/\kappa_K) \to 1, \tag{T.6}$$

where $\kappa_K \simeq \mathbb{F}_p$ and $\kappa_{\overline{K}} \simeq \overline{\mathbb{F}}_p$ denote the residue fields of K and $\overline{K}$, respectively.

Since we are dealing with an elliptic curve, we can exploit the fact that it comes with a group law. Indeed, from our definition in equation (T.1), we see that the elliptic curve forms a group under multiplication inherited from $\overline{K}^\times$. It turns out that the cohomology group $H^1(E_{\overline{K}}, \mathbb{Z}_\ell)$ is dual to $H_1(E_{\overline{K}}, \mathbb{Z}_\ell)$, which can in turn be written as the inverse limit on n:

$$H_1(E_{\overline{K}}, \mathbb{Z}_\ell) = \varprojlim E_{\overline{K}}(\ell^n). \tag{T.7}$$

Here, $E_{\overline{K}}(N)$ is interpreted as a multiplicative subgroup of $E_{\overline{K}} = \overline{K}^\times / q^{\mathbb{Z}}$ defined by the conditions:

$$E_{\overline{K}}(N) = \left\{ e \in E_{\overline{K}} \quad \text{such that} \quad e^N = 1 \right\}, \tag{T.8}$$

namely the N-torsion points of the elliptic curve. The inverse limit involves the further refinement that we work with $N = \ell^n$. So, to track the action of monodromy, it is enough to study how it acts on the space of N-torsion points.

Our plan in this Appendix will be to carry out this computation in a "leading order" approximation for a specific set of illustrative choices. To this end, we fix $p = 3$, $\ell = 2$, $n = 2$ (so that $N = 4$), and $q = 3$. Moreover, rather than work with the full separable closure of $K = \mathbb{Q}_3$, we shall instead confine our analysis to the finite extension $L = \mathbb{Q}_3(\omega, i)$, where $\omega^3 = 1$ and $i^4 = 1$. Our first task is to compute the various terms in the short exact sequence of line (S.10):

$$1 \to I_{L/K} \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(\kappa_L/\kappa_K) \to 1. \tag{T.9}$$

First, we calculate $\mathrm{Gal}(L/K)$. The Galois group is generated by the transformations $\omega \mapsto \omega^2$ and $i \mapsto -i$, and we have:

$$\mathrm{Gal}(L/K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \tag{T.10}$$

in the obvious notation.

Next, we calculate $\mathrm{Gal}(\kappa_L/\kappa_K)$. Now, recall that in $\mathbb{Q}$, $(1-\omega)^2$ is ramified at 3. Letting $\nu = (1-\omega)$, we see that it is enough to compute the residue field of L at place $(1-\omega)$, but this just reduces to "dropping" the contribution from ω altogether:

$$\mathrm{Gal}(\kappa_L/\kappa_K) = \mathrm{Gal}(L_\nu/\kappa_K) \simeq \mathrm{Gal}(\mathbb{F}_3(i)/\mathbb{F}_3). \tag{T.11}$$

Note, however, that under Frobenius conjuagation, we have $i \mapsto i^3 = -i$, so $\mathrm{Gal}(\mathbb{F}_3(i)/\mathbb{F}_3) \simeq \mathbb{Z}/2\mathbb{Z}$ is just one of the factors appearing in equation (T.10). Returning to the form of our short exact sequence, we conclude that the inertia group is given by:

$$I_{L/K} = (\omega \mapsto \omega^2) \simeq \mathbb{Z}/2\mathbb{Z}. \tag{T.12}$$

We now ask how the inertia group $I_{L/K}$ acts on the Tate module $H_1(E_{\overline{K}}, \mathbb{Z}_\ell) = \lim\limits_{\leftarrow} E_{\overline{K}}(\ell^n)$, at the level of approximation already mentioned. In particular, we just consider the action of $I_{L/K}$ on the $N = 2^2 = 4$-torsion points of E_L, i.e.:

$$I_{L/K} : E_L(N) \to E_L(N). \tag{T.13}$$

where we view $E_L(N)$ as a module over $\mathbb{Z}/N\mathbb{Z}$. For 4-torsion points, we claim that such a basis is provided by:

$$b_1 = \frac{1-\omega}{1+\omega} \quad \text{and} \quad b_2 = i\,. \tag{T.14}$$

First of all, we observe that $i^4 = 1$, so this means b_2 is a 4-torsion point.

Next, consider b_1. Observe that $(b_1)^2$ is actually an element of K, since under the action $\omega \mapsto \omega^2$ of $\mathrm{Gal}(L/K)$, we have:

$$(b_1)^2 = \left(\frac{1-\omega}{1+\omega}\right)^2 \mapsto \left(\frac{1-\omega^{-1}}{1+\omega^{-1}}\right)^2 = \left(\frac{\omega-1}{\omega+1}\right)^2 = (b_1)^2\,. \tag{T.15}$$

In fact, we also have:

$$\left(\frac{1-\omega}{1+\omega}\right)^2 = -3, \tag{T.16}$$

so in particular, we have that b_1 satisfies:

$$(b_1)^4 = 9 \in K \tag{T.17}$$

Since we are working in the quotient group $E_L = L^\times/q^{\mathbb{Z}}$, we see that for $q = 3$, $(b_1)^4 = 1$ in E_L.

To complete the discussion, we finally consider the action of the inertia group $I_{L/K} \simeq (\omega \mapsto \omega^2)$ on $E_L(N)$. As already mentioned, $b_2 \mapsto b_2$. As for b_1, we have $b_1 \mapsto -b_1 = b_1(b_2)^2$. Writing this out as an additive group law (rather than the multiplicative group action inherited from $L^{\times}$), we can instead write this as:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{T.18}$$

which we recognize as the monodromy associated with an I_2 singularity of an elliptic fibration! Observe that because we are just working with 4-torsion points (instead of the full inverse limit), then we return to the original basis vector after two applications of the inertia group (i.e., since $4 \equiv 0 \bmod 4$). As we already remarked in section 17, this is very suggestive, and strongly suggests that similar monodromic structure will persist in the full action of the inertia group $I_{\overline{K}/K}$, as well as as for different choices of the underlying parameters such as p, ℓ and q.

U Aspects of Berkovich Spaces

In this Appendix we present a brief account of the Berkovich spectrum. For further details, we refer the interested reader to [32] as well as the readable account in [263]. Again, in keeping with the informal flavor of these notes, we shall appeal to [359] for our account. To make these notions a bit more concrete, we then present the main examples discussed in the main body of these notes, involving $\mathbb{A}^1_{\text{Berk}}$, $\mathbb{P}^1_{\text{Berk}}$ and $\mathbb{H}_{\text{Berk}}$.

To begin we recall that for a ring A, we can specify a seminorm as a map $|\cdot| : A \to \mathbb{R}_{\geq 0}$ satisfying the conditions:

$$|0| = 0 \tag{U.1}$$

$$|1| = 1 \tag{U.2}$$

$$|f + g| \leq |f| + |g| \tag{U.3}$$

$$|fg| \leq |f|\,|g|\,. \tag{U.4}$$

We say the seminorm is multiplicative if $|fg| = |f|\,|g|$, and if $|f| = 0$ implies $f = 0$, we say that we have a norm. Assuming A is a ringed norm space with norm $\|\cdot\|$, then we can construct the Berkovich spectrum $\mathcal{M}(A)$ as given by the space of multiplicative seminorms bounded by the norm.

In general, the Berkovich topology is the weakest one such that for any $f \in A$, the map:

$$\varphi_f : \mathcal{M}(A) \to \mathbb{R}_{\geq 0} \tag{U.5}$$

$$|\bullet| \mapsto |f| \tag{U.6}$$

is continuous.

At this point, it may appear somewhat jarring to formulate the relevant points in terms of the "space of seminorms," but we can see how the standard notions of points can be associated to each such seminorm. To illustrate, we now fix K and consider a ball of radius r centered at a point $a \in K$, which we define as:

$$B_{(a,r)} = \{z \in K \quad \text{such that} \quad |z - a| \leq r\}\,. \tag{U.7}$$

Now, the point is that for each such ball, we can define a corresponding seminorm on $K[x]$. To see how this comes about, we simply define this seminorm $|\cdot|_{B_{(a,r)}}$ by specifying its values on all $f \in K[x]$ by the condition:

$$|f|_{B_{(a,r)}} = \sup_{z \in B_{(a,r)}} |f(z)|\,. \tag{U.8}$$

Observe that the special case where $r = 0$ gives us back just the usual points of K, but that now we also include finite sized disks which also include this initial point. In fact, Berkovich's theorem tells us that every point $x \in \mathbb{A}^1_{\text{Berk}}$ can be viewed as a nested sequence of disks $B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq ... \supseteq B(a_n, r_n) \supseteq ...$ which conveges in the sense that (see e.g., [32] as well as Theorem 1.3.1 of reference [263]):

$$|f|_x = \lim_{n \to \infty} |f|_{B(a_n, r_n)} \,. \tag{U.9}$$

Further, two such nested sequences define the same point if and only if each intersection:

- Case a: Each has a nonempty intersection, and their intersections are the same;

or:

- Case b: Both have empty intersection, and the sequences are cofinal.[135]

So in other words, we can associate to each point x a corresponding intersection $B = \cap B(a_n, r_n)$. In this case, the points of the affine line split into four distinct types. These are given by:

- Type I: B is a point of K
- Type II: B is a closed disk with radius belonging to $|K^\times|$
- Type III: B is a closed disk with radius not belonging to $|K^\times|$
- Type IV: $B = \varnothing$

So, in particular, we get the points of K, but also more. In the above, $|K^\times|$ just denotes the possible norms of elements of K^*.

By a similar token, we can extend the Proj construction of algebraic geometry to append a point at infinity. This gives us $\mathbb{P}^1_{\text{Berk}}$. Lastly, we can speak of $\mathbb{H}_{\text{Berk}} = \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1_{\text{Berk}}(K)$ which we can think of as a "hyperbolic space". We refer to $\mathbb{H}^{\mathbb{R}}_{\text{Berk}}$ as the space of type II or III points in $\mathbb{H}_{\text{Berk}}$, and $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}$ as the space of type II points in $\mathbb{H}_{\text{Berk}}$.

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[135]Recall that for a subset $B \subseteq A$ of a preordered set $(A, \leq)$,[136]we say that B is cofinal with respect to $\leq$ when, for every $a \in A$, there exists a $b \in B$ such that $a \leq b$. In the case at hand, we take B and A to be the sets specified by the corresponding sequences, and $\leq$ is specified by the containment relation $\subseteq$ of the disks.

[136]In a preordered set $(A, \leq)$ we have, $a \leq a$ for all $a \in A$ (reflexivity) and $a \leq b$ and $b \leq c$ implies $a \leq c$ for $a, b, c \in A$. Having a partial order involves the slightly stronger condition that $a \leq b$ and $b \leq a$ implies $a = b$.

V Tropicalization

In this Appendix we briefly summarize some aspects of tropicalization of a p-adic analytic space. To set the stage, we recall that in the physics literature, tropical geometry often appears in the study of (p, q) webs of branes, and their close connection to toric geometry (see e.g., [360, 361]). Here we wish to discuss the case of a p-adic analytic space X_{an} and its associated tropical geometry $\text{Trop}(X_{\text{an}})$.

Our discussion follows that in [267]. To begin, we recall that for a p-adic variety X over a field k, we can specify a moment map to a d-dimensional split algebraic torus T via a morphism:[137]

$$\mu : X \to T, \tag{V.1}$$

Under analytification, we then get a moment map:

$$\mu_{\text{an}} : X_{\text{an}} \to T_{\text{an}}. \tag{V.2}$$

We can specify a tropical map for our analytic torus as follows (with a small abuse of notation):

$$\text{Trop: } T_{\text{an}} \to \mathbb{R}^d \tag{V.3}$$

$$(z_1, ..., z_n) \mapsto (-\log|z_1|, ..., -\log|z_d|), \tag{V.4}$$

where the z_i denote local coordinates on the torus, i.e., writing it as the group scheme $T = \text{Spec}_k\, k[T_1, T_1^{-1}, ..., T_d, T_d^{-1}]$, we compose with a suitable evaluation map. We denote by $T_{\text{trop}} = \text{Trop}(T_{\text{an}})$ the image set of the tropical map. Then, using our moment map μ_{an}, we see that we can extend this to a tropicalization map $\mu_{\text{trop}} : X_{\text{an}} \to \mathbb{R}^n$, and specify the image as $\text{Trop}(X_{\text{an}}) = X_{\text{trop}}$.

This tropicalization serves as the "skeleton" for the analytic space, and moreover, there is a natural section and embedding available. To illustrate, the section $s : T_{\text{trop}} \to T_{\text{an}}$ is given by specifying, for every $(t_1, ..., t_d) \in T_{\text{trop}}$, a seminorm as implicitly defined by the condition that for every $\varphi \in k[T_1, T_1^{-1}, ..., T_d, T_d^{-1}]$ written as:

$$\varphi = \sum_m \varphi_m T^m, \tag{V.5}$$

we have:

$$|\varphi(s(t_1, ..., t_d))| = \max_{m \in \mathbb{Z}^n} |\varphi_m| \exp(-m_1 t_1 + ... - m_d t_d). \tag{V.6}$$

In particular, $s(0)$ is the usual "Gauss point" of T_{an}. This also means there is a natural notion of the skeleton X_{trop} embedding in X_{an}.

[137]Namely, we introduce the ring $R = k[T_1, T_1^{-1}, ..., T_n, T_n^{-1}]$, and we identify T with the group scheme $\text{Spec}_k R$. In the more common setting of a toric variety, this would just be identified with the standard complex torus $(\mathbb{C}^\times)^n$ which acts on the coordinates of a toric variety.

Let us also mention that we can view Berkovich space as the inverse limit of all possible tropicalizations. This is demonstrated in reference [248] (see also [362]).

It is interesting to consider the tropicalization map as applied to some affine curves. To this end, consider the case of the hyperelliptic curve Σ over $\mathbb{C}_p$ embedded in affine space $\mathbb{A}^2$:

$$y^2 = x^N + c_{N-1}x^{N-1} + ... + c_0 = \prod_i (x - e_i), \tag{V.7}$$

where the $e_i \in \mathbb{C}_p$ are just the roots of our degree N polynomial in x. To construct the image set $\Sigma_{\text{trop}}(\mathbb{C}_p)$, we first observe that:

$$-\log|y| = -\left(\log|x - e_1| + ... + \log|x - e_N|\right). \tag{V.8}$$

So in particular, we see that there are distinguished rays for each $|x - e_i| \to 0$ for $i = 1, ..., N$ which tend to $(-\log|e_i|, +\infty)$ in $\mathbb{R}^2$. These are all connected up along a graph to the additional ray $(-\infty, -\infty)$ as associated with the region $(x, y) \to (\infty, \infty)$, the "point at infinity" in $\mathbb{A}^2$.

W Evaluating a Berkovich Amplitude

In this Appendix we evaluate some p-adic amplitudes. One interest will be in the limit $n \to \infty$, which we view as providing an increasingly accurate estimate with the worldsheet given by Berkovich space. For $\mathbb{Q}_q$ some degree n unramified extension of $\mathbb{Q}_p$, we will be interested in evaluating the integral:

$$B_n(a,b) = \int_{\mathbb{Q}_p} dx \;\; |x|^{a-1} \, |1-x|^{b-1} \, , \tag{W.1}$$

for $a, b \in \mathbb{C}$, where here, the measure factor is just the expected one for $\mathbb{Q}_q$, viewed as an n-dimensional vector space over $\mathbb{Q}_p$, and we have also extended the norm, as appropriate. We shall aim to express this in terms of suitable combinations of the p-adic Gamma function, defined as:

$$\Gamma_p(s) = \int_{\mathbb{Q}_p} dx \;\chi(x)\, |x|^{s-1} \, , \tag{W.2}$$

for $s \in \mathbb{C}$, where $\chi : \mathbb{Q}_p \to \mathbb{C}$ is the character on the additive group $(\mathbb{Q}_p, +)$. We remark that in the p-adic setting, we have:

$$\Gamma_p(s) = \frac{1 - p^{s-1}}{1 - p^{-s}}. \tag{W.3}$$

Additionally, we have the identity:

$$\Gamma_p(s)\Gamma_p(1-s) = 1. \tag{W.4}$$

There is also the identity:

$$|x|^{a-1} = \Gamma_p\left(a\right) \int_{\mathbb{Q}_p} du \;\chi(ux)\, |u|^{-a} \, , \tag{W.5}$$

as well as its generalization to $\mathbb{Q}_q$. For additional relations which are closely related, see for example reference [272] and references therein.

We begin by reviewing the "standard case" of $\mathbb{Q}_p$ and we then turn to a similar evaluation for a degree n unramified extension $\mathbb{Q}_q$.

W.1 The Case $\mathbb{Q}_p$

Let us begin with an evaluation of the "standard case," as obtained by evaluating equation (W.1) in the special case $\mathbb{Q}_q = \mathbb{Q}_p$. Using our relations for the Gamma function, we can first write:

$$B(a,b) = \int\limits_{\mathbb{Q}_p} dx \ \ |x|^{a-1} \, |1-x|^{b-1} \tag{W.6}$$

$$= \Gamma_p\,(a)\,\Gamma_p\,(b) \int\limits_{\mathbb{Q}_p} dxdudv \ \ \chi(ux + v(1-x))\, |u|^{-a} \ \ \ |v|^{-b} \tag{W.7}$$

$$= \Gamma_p\,(a)\,\Gamma_p\,(b) \int\limits_{\mathbb{Q}_p} dxdudv \ \ \chi(x(u-v) + v)\, |u|^{-a} \ \ \ |v|^{-b} \tag{W.8}$$

or:

$$B(a,b) = \Gamma_p\,(a)\,\Gamma_p\,(b) \int\limits_{\mathbb{Q}_p} dxdudv \ \ \chi(x(u-v))\chi(v)\, |u|^{-a} \ \ \ |v|^{-b}. \tag{W.9}$$

Next, perform the integral over x. This leaves us with a Dirac delta function, evaluated over the p-adics:

$$B(a,b) = \Gamma_p\,(a)\,\Gamma_p\,(b) \int\limits_{\mathbb{Q}_p} dudv \ \ \delta(u-v)\chi(v)\, |u|^{-a} \ \ \ |v|^{-b} \tag{W.10}$$

$$= \Gamma_p\,(a)\,\Gamma_p\,(b)\,\Gamma_p(c) \quad \text{with} \quad a+b+c=1. \tag{W.11}$$

We remark that this is just the formula obtained in [35], which exhibits the necessary crossing symmetries expected of the p-adic string amplitude.

W.2 The Case $\mathbb{Q}_q$

Consider next the case of an unramified extension of $\mathbb{Q}_p$. There is an analagous set of manipulations we can perform in this case, which relies on used the q-extension of the Gamma function given by:

$$\Gamma_q(s) = \frac{1-q^{s-1}}{1-q^{-s}}. \tag{W.12}$$

In particular, we have, for the degree n field extension:

$$B_n(a,b) = \int\limits_{\mathbb{Q}_q} dx \ \ |x|^{a-1} \, |1-x|^{b-1} = \Gamma_q\,(a/n)\,\Gamma_q\,(b/n)\,\Gamma_q(c/n) \quad \text{with} \quad a+b+c=1, \tag{W.13}$$

the only change being that we replaced p with q and rescaled a, b, c. More explicitly, we have:

$$B_n(a,b) = \frac{1-p^{a-n}}{1-p^{-a}}\frac{1-p^{b-n}}{1-p^{-b}}\frac{1-p^{c-n}}{1-p^{-c}} \quad \text{with} \quad a+b+c=1. \tag{W.14}$$

And so, in the large n limit, we get:

$$\lim_{n\to\infty} B_n(a,b) = \frac{1}{1-p^{-a}}\frac{1}{1-p^{-b}}\frac{1}{1-p^{-c}} \quad \text{with} \quad a+b+c=1. \tag{W.15}$$

Based on this, we make the further assertion that on $\mathbb{A}^1_{\mathrm{Berk}}$, we have:

$$\mathfrak{B}(a,b) = \int\limits_{\mathbb{A}^1_{\mathrm{Berk}}} dx\ \ |x|^{a-1}\,|1-x|^{b-1} = \frac{1}{1-p^{-a}}\frac{1}{1-p^{-b}}\frac{1}{1-p^{-c}}. \tag{W.16}$$

W.3 Reinterpretation as a Contour Integral

In section 20 we proposed a contour integral prescription for evaluating open string amplitudes in Berkovich spaces. Here we show how to recast the standard integral over $\mathbb{Q}_p$, as an approximation of a contour-like integral. To begin, we start with:

$$B(a,b) = \int\limits_{\mathbb{Q}_p} dx\ \ |x|^{a-1}\,|1-x|^{b-1}. \tag{W.17}$$

We break up this integral into shells of fixed $|x| = p^m$. Observe also that the "volume" of the ball $\mathbb{B}_m$ with $|x| \leq p^m$ is:

$$\mathrm{Vol}(\mathbb{B}_m) = \int\limits_{\mathbb{B}_m} dx = p^m, \tag{W.18}$$

so the volume of S_m, a fixed radius shell with $|x| = p^m$ is then:

$$\mathrm{Vol}(S_m) = \int\limits_{S_m} dx = p^m - p^{m-1}. \tag{W.19}$$

With this in place, we observe that we can split up the sum into the contributions $|x| < 1$, $|x| = 1$, and $|x| > 1$:

$$\begin{aligned} B(a,b) &= \int\limits_{\mathbb{Q}_p} dx\ \ |x|^{a-1}\,|1-x|^{b-1} && \text{(W.20)}\\ &= \int\limits_{|x|<1} dx\ \ |x|^{a-1}\,|1-x|^{b-1} && \text{(W.21)}\\ &+ \int\limits_{|x|=1} dx\ \ |x|^{a-1}\,|1-x|^{b-1} && \text{(W.22)}\\ &+ \int\limits_{|x|>1} dx\ \ |x|^{a-1}\,|1-x|^{b-1}. && \text{(W.23)} \end{aligned}$$

For $|x| \neq 1$, each integral can in turn be broken up into fixed radius shells:

$$\int\limits_{|x|<1} dx \;\; |x|^{a-1} \, |1-x|^{b-1} = \sum_{m<0} \mathrm{Vol}(S_m)(p^m)^{a-1} = -\frac{1-p^{-1}}{1-p^a} \tag{W.24}$$

$$\int\limits_{|x|>1} dx \;\; |x|^{a-1} \, |1-x|^{b-1} = \sum_{m>0} \mathrm{Vol}(S_m)(p^m)^{a+b-2} = -\frac{1-p^{-1}}{1-p^{a+b-1}}. \tag{W.25}$$

The evaluation for the shell $|x| = 1$ is conveniently handled by decomposing $x = z_0 + y$, with $z_0 \in \{0, ..., p-1\}$ and $|y| < 1$. In this case, then, we can write:

$$\int\limits_{|x|=1} dx \;\; |x|^{a-1} \, |1-x|^{b-1} = \sum_{z_0=0}^{p-1} \int\limits_{|y|<1} dy \;\; |1 - z_0 - y|^{b-1}, \tag{W.26}$$

which again can be split up into the contributions from $z_0 \neq 1$ and $z_0 = 1$ as:

$$\int\limits_{|x|=1} dx \;\; |x|^{a-1} \, |1-x|^{b-1} = (p-2) \int\limits_{|y|<1} dy + \int\limits_{|y|<1} dy \;\; |y|^{b-1} \tag{W.27}$$

$$= (p-2)p^{-1} - \frac{1-p^{-1}}{1-p^b}. \tag{W.28}$$

Before proceeding further, let us verify that this produces the expected formulae for the standard p-adic open string. We indeed find:

$$B(a,b) = \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}, \tag{W.29}$$

with $a + b + c = 1$.

We would now like to interpret this as a "contour integral". To this end, introduce:

$$x_m = p^m, \quad y_m = p^m \tag{W.30}$$

as well as the finite difference:

$$\Delta x_m = p^m - p^{m-1}, \quad \Delta y_m = p^m - p^{m-1}. \tag{W.31}$$

For the regions $|x| < 1$ and $|x| > 1$ we can equally well write:

$$\sum_{m \in \mathbb{Z}} \Delta x_m \, |x_m|^{a-1} \, |1 - x_m|^{b-1} = \int\limits_{|x|<1} dx \;\; |x|^{a-1} \, |1-x|^{b-1} + \int\limits_{|x|>1} dx \;\; |x|^{a-1} \, |1-x|^{b-1}. \tag{W.32}$$

For the region $|x| = 1$, more care is required because, as we have seen, it is simplest to split the contributions up as $x = z_0 + y$ with $z_0 = \{0, ..., p-1\}$ and $|y| < 1$. In this case, we have:

$$\sum_{m<1} \Delta y_m \left|y_m\right|^{b-1} + (p-2) \int\limits_{|y|<1} dy = \int\limits_{|x|=1} dx \;\; |x|^{a-1} \left|1-x\right|^{b-1}. \tag{W.33}$$

So, at least as this level, the discretized sum over the x_m and the y_m is quite close to that of the p-adic Euler Beta function:

$$B(a,b) = \sum_{m\in\mathbb{Z}} \Delta x_m \left|x_m\right|^{a-1} \left|1-x_m\right|^{b-1} + \sum_{m<1} \Delta y_m \left|y_m\right|^{b-1} + (p-2) \int\limits_{|y|<1} dy. \tag{W.34}$$

The first two terms appear as "contours" in the sense that the one over x_m is along the analog of the radial line, and the one over y_m encircles the region $|x| = 1$. This leaves us with the remainder term over $|y| < 1$, the open ball. Let us also note that this contribution is independent of a and b. So in other words, at least as far as the momentum dependence of the amplitude goes, for both the x_m integral and the y_m integral, there is a local ordering (as indexed by $m \in \mathbb{Z}$), which we can write as:

$$B(a,b) = \int\limits_{\gamma_{\mathrm{rad}}} dz \;\; |z|^{a-1} \left|1-z\right|^{b-1} + \int\limits_{\gamma_1} dz \;\; |z|^{a-1} \left|1-z\right|^{b-1} + ... \tag{W.35}$$

where the "..." refers to terms independent of a and b.

Let us now generalize this sort of integral. Along these lines, we now consider the integral over $x = t_{N+1}$:

$$B(a_1, ..., a_N) = \int\limits_{\mathbb{Q}_p} dx \;\; |x-t_1|^{a_1-1} \left|x-t_2\right|^{a_2-1} ... \left|x-t_N\right|^{a_N-1}. \tag{W.36}$$

Suppose first that the t_i are distinct. In this case we can order the t_i according to their norms, i.e., $|t_1| < ... < |t_N|$, and then we can perform a similar decomposition of the x integral into the regions $|x| < |t_1|$, $|x| = |t_1|$,..., $|x| = |t_N|$ and $|x| > |t_N|$. In this case, we again see that there is a quite similar decomposition in terms of contour integrals, one which runs along the "radial axis", and one for each of the t_i. So in other words, at least as far as the a_i dependence is concerned, we can write a formal "contour":

$$\gamma = \gamma_{\mathrm{rad}} + \gamma_1 + ... + \gamma_N, \tag{W.37}$$

and then we can write this as:

$$B(a_1, ..., a_N) = \int\limits_{\gamma} dz \;\; |z-t_1|^{a_1-1} \left|z-t_2\right|^{a_N-1} ... \left|z-t_N\right|^{a_N-1} + ..., \tag{W.38}$$

where again, the "..." is independent of the a_i's.

Suppose next that some of the t_i have the same norm, for example $|t_i| = |t_{i+1}|$ for some i, with the same ordering otherwise. Now when we evaluate our integral over x, the "residue" picked up at this circle involves a contour $\gamma_{i,i+1}$, as associated with $t_i + t_{i+1}$. More generally, if we have two or more t_i of the same norm, we can introduce the contour $\gamma_{i_1,\ldots,i_l}$ as the one associated with the point $t_{i_1} + \ldots + t_{i_l}$. Continuing in this way, we can again present a contour integral formula interpretation.

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